MODEL OF COMPOSITE DEGRADATION DUE TO FATIGUE DAMAGE

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ABSTRACT

This report first reviews the problems of developing fatigue models for unidirectional composites and then describes a new model that aims to predict the cycle-dependent strength of a unidirectional fibre reinforced composite arising from axial fatigue loading. The model assumes that microstructural damage leads to interfaces that are very weak so that a parallel two bar model can be developed where one bar represents the behaviour of the fibres which are regarded as a loose bundle, and where the second bar represents the matrix. The principal effect of fatigue loading is assumed to be the cycle dependent strength degradation of individual fibres arising from microscopic failure mechanisms within the fibres. The initial strength of the fibres is assumed to be governed by the Weibull distribution which, when combined with a fibre failure criterion, defines an initial statistical distribution of fibre strength characteristics. Composite degradation arising from fatigue loading is assumed to be governed by a damage growth law. The model developed here parallels, but in a more general manner, a recently developed model of composite degradation arising from environmental exposure.

It has been shown that, when the residual strength of the composite is divided by its static strength, the resulting strength ratio is virtually independent of the matrix properties. The strength ratio for a unidirectional composite, where fibres are prone to fatigue damage, is thus predictable from the static strength of the composite, and the cycle dependence of the residual strength of a bundle of loose fibres. The corollary to this is that the fatigue performance of unidirectional composites might be estimated experimentally from the use of static strength data, and an examination of the fatigue behaviour of a bundle of loose fibres.

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Important notation

F	Maximum load applied during fatigue loading
F_{min}	Minimum load applied during fatigue loading
ΔF	Load range for uniform amplitude fatigue loading
R	Fatigue load ratio (F/ F_{min})
N	Number of fatigue cycles
N_f	Number of cycles to failure due to fatigue loading
$S(N)$	Residual strength after N fatigue cycles
A	Area of cross section of a single fibre
V_f	Fibre volume fraction
V_m	Matrix volume fraction
$F_b(N)$	Maximum load applied to the fibre bundle
$\sigma(N)$	Maximum stress in a surviving fibre
$\Delta\sigma(N)$	Stress range in a surviving fibre
$\sigma_b(N)$	Effective stress applied to the bundle of fibres
$\sigma_{app}(N)$	Effective stress applied to the composite
$\sigma_m(N)$	Maximum stress in the matrix
$\Delta\sigma_m(N)$	Stress range in the matrix
$\varepsilon(N)$	Axial strain in fibres and matrix after N fatigue cycles
E_f	Young's modulus of the fibres
E_m	Young's modulus of the matrix
$E_b(N)$	Effective modulus of the fibre bundle
$E_c(N)$	Effective modulus of the composite
$P(N)$	Number of surviving fibres after N fatigue cycles
α	Dimensionless parameter defined by V_mE_m / V_fE_f
a	Parameter characterising fibre damage due to fatigue loading
$a_c(N)$	Critical value of a after N cycles
$X_o(N)$	Initial value of a that requires N cycles to reach the critical value $a_c(N)$
$\sigma_i(N)$	Initial strength of fibres that fail after N cycles
m	Weibull exponent
σ_o	Scaling stress used in Weibull distribution
C	Coefficient appearing in the fibre degradation law
q, r	Exponents appearing in the fibre degradation law
$\hat{x}$	Normalised value of any quantity x

1. SETTING THE SCENE FOR MODEL DEVELOPMENT

The fatigue performance of all materials is usually characterised by S-N data that relates the maximum stress applied during uniform amplitude fatigue loading to the lifetime, which is the number of fatigue cycles to failure of a virgin testpiece. Fatigue performance is also affected by the mean stress that is applied during loading. A useful type of design data concerns the dependence of the residual strength of the material on the number of cycles to which the testpiece has been subjected. For monolithic materials the mechanism of fatigue damage is usually the formation of micro-cracks that gradually increase in size during fatigue loading until the largest crack is long enough to cause catastrophic failure of the testpiece. Such microcracks can initiate on the testpiece surface as a result of machining damage or, in the case of carefully prepared testpieces, as a result of localised plastic flow at the free surface leading to the development of extrusions and intrusions from which cracks eventually form. Microcracks can also form internally at metallurgical defects that cause localised stress concentrations which are large enough to enable reversed plastic flow mechanisms to operate during fatigue loading leading to the development of microcracks. Stress concentrations arising from specimen end effects can also initiate microcracks during fatigue loading. Key aspects of such microstructural damage growth are the fatigue growth law that determines the rate of growth of the defects, and the fracture toughness of the material that controls the occurrence of the catastrophic failure event. Also key and exceedingly difficult to model is the damage initiation stage of the fatigue process. It will become clear that the problem of modelling fatigue damage in monolithic materials is much simpler than the modelling required to predict the fatigue performance of unidirectional fibre reinforced composites, and yet there are very significant problems associated with fatigue damage prediction in monolithic materials. There is no generally accepted approach to the predictive modelling of S-N fatigue data, principally because of the problems of predicting how many fatigue cycles are needed to initiate a micro-crack whose size is larger enough for growth under fatigue loading to be predictable using convention fracture mechanics based on a fatigue crack growth law.

Turning now to the fatigue performance of unidirectional composites, the approach has been to characterise the fatigue performance of composite materials using exactly the same type of test that is used for monolithic materials, i.e. the measurement of S-N data and residual strength as a function of the number of fatigue cycles that have been applied. A very useful review by Konur & Matthews [1] includes an examination of the effect of both fibre and matrix properties on the fatigue performance of unidirectional composites. It is clear from this review that the micro-mechanisms of fatigue damage development are very complex and the observed behaviour of composite fatigue performance is very unlikely to be represented adequately by a single mechanistic model. This presents severe problems when attempting to develop a predictive model of behaviour for the fatigue loading of unidirectional fibre reinforced materials. The following discussion attempts to develop a rational approach to modelling fatigue behaviour of these materials.

The definition of S-N data requires a precise definition of the point of failure. For monolithic materials failure occurs when the testpiece breaks into two pieces provided that the failure occurs within the gauge length of the testpiece. When testing unidirectional composites such a clear point of failure rarely occurs. If it does such failures are often a result of the presence of unrepresentative defects in the material, or of machining damage. A testpiece made of well machined good quality material will often fail as a result of fatigue damage entering the grip region by a splitting mechanism. The testpiece fails because of such events which might not be relevant to the failure of engineering components, e.g. fibre wound pressure vessels where gripping aspects are not relevant. S-N data from testpieces have little relevance to fatigue performance of the pressure vessel. The uncertainty of the relevance of S-N fatigue test data to the design of engineering components is a major obstacle to future developments of the subject. The approach to be used here is to consider S-N data as though it had been collected from a perfect test procedure and machine, where there is negligible effect of gripping (including friction) and specimen geometry on the fatigue failure, and negligible self-heating arising from energy dissipation in the testpiece. This obviously will present major problems of validation of the model by experimental measurement.

Given that a perfect testing procedure is available such that a definite point of testpiece failure can be defined that is not influenced by gripping factors, the point of failure of a unidirectional composite is regarded to be the occurrence of a catastrophic failure that is characterised by the failure of all the fibres in the composite, and of the matrix. This definition raises the issue as to whether individual fibres are prone to fatigue damage, i.e. if fibres were fatigue tested in a single fibre test, would they exhibit an S-N curve and show a cycle dependent residual strength? There does not seem to be a definitive answer to this question. The approach to be adopted here is to assume that fibre strength degrades as a result of fatigue loading. One rationale for doing this, of relevance to later tasks in the Project CPD2, is based on the fact that environmental exposure can certainly degrade fibre properties (e.g. glass fibres) when subject to static loading, and that such degradation is likely to accelerate when the fibres are also subject to fatigue loading at some specified level of mean load.

Having discussed the difficulties of defining failure for unidirectional composites, and the consequent effect on the S-N data itself, it is now appropriate to consider the micro-mechanisms that can occur within unidirectional composites during fatigue loading. It is assumed at the start of fatigue testing that the composite is such that the matrix is well bonded to all the fibres. During fatigue loading microstructural damage can develop in various forms each having different consequences on subsequent damage development and fatigue performance. If the Poisson's ratio of fibre and matrix, and thermal stresses, are such that during cycling the fibre/matrix interface experiences some tensile normal tractions it is possible for interface defects to grow in size developing fibre/matrix debonds. In principle all fibre/matrix interfaces could become debonded by this mechanism. On the other hand if the interfacial normal tractions are always compressive the interface will remain bonded provided no other damage mechanism influences the interface such as a matrix crack or fibre failure. For polymer based composites it is often damage in the matrix that affects fatigue performance. Here defects within the matrix will grow in size as a result of the cyclic stress experienced locally within the matrix. The tips of such microcracks can grow towards the fibres and the fibre/matrix interfaces. Load transfer locally arising from the presence of the matrix crack can lead to a local stress concentration in the fibre causing a fibre failure. Such behaviour is expected for well bonded fibres and matrix. The occurrence of a fibre fracture can then promote fibre/matrix debonding in the neighbourhood of the fibre fracture. If the composite were such that the fibres and matrix are not well bonded then the fibre/matrix interface might debond before fatigue loading results in the matrix crack reaching the interface. The debonded interface prevents the build up of a large stress concentration and fibre failure.

Fibre/matrix debonding, however caused, can promote the occurrence of longitudinal splits which grow in length, often into the gripping region of testpieces, thus causing premature failure of the composite. Such a failure mechanism must be regarded as a feature of the test method and S-N data based on such failures will not necessarily be representative of the fatigue performance of the material in more favourable situations (e.g. fatiguing of fibre-wound tubes).

Having dealt with matrix fatigue mechanisms it is now appropriate to consider fatigue mechanisms in fibres. Consider first of all the case where either the matrix is not susceptible to fatigue damage, or is such that fatigue damage in the matrix does not lead to stress concentration in the fibre because of fibre/matrix debonding, or because the matrix is so degraded by fatigue that it is incapable of transferring load to the fibres. If fibres do not degrade as a result of fatigue loading then if the unidirectional composite is loaded so that the maximum stress in the fibres is less than their in situ strength, the composite will not fail. If fibres do degrade as a result of fatigue loading then the performance of the composite is determined principally by fibre behaviour. This situation is mathematically tractable and will be the basis of the model to be developed. The strength reduction of the fibres can result because of the progressive growth of fibre defects caused by fatigue crack growth at a microscopic level. For fibres subjected to fatigue loading the failure mechanisms are not always so clearly defined and it becomes necessary to consider the concept of an effective fibre strength characteristic that has the role of relating the fibre strength to a measure of its state of degradation.

Prolonged exposure to fatigue loading can lead to a deterioration in interface properties. Given that the axial strength of a unidirectional composite is not affected to a great extent by interface properties, it is reasonable to assume, when modelling the axial behaviour of a unidirectional composite, that the interfaces in the composite following prolonged fatigue exposure have no 'strength'. This enables a relatively simple approach to be taken that should provide good insight into the axial behaviour of a composite when exposed to fatigue loading.

Because of the dominance of fibre behaviour, earlier modelling work applied to fibre composites [2, 3] regarded the unidirectional composite as a loose bundle of parallel fibres having equal length, so that the relatively low load carrying capacity of the matrix was ignored. The fibres in the bundle were assumed to be attached to rigid supports which were able to share the applied load equally between all surviving fibres. The objective of this report is to use the loose bundle model, taking the load carrying capacity of the matrix into account, when considering the axial behaviour of unidirectional fibre reinforced composites subject to axial fatigue loading. In addition, the model is extended so that a more general fibre damage growth law can be used. The procedure has already been carried out for the case of composite degradation arising from environmental exposure [4] for the special case where fibre degradation arises because of the growth of defects governed by the laws of fracture mechanics. The advantage of adopting this approach is that in a later stage of the project the models developed can be further developed in order to examine the effects of combined environmental and fatigue loading on the performance of a unidirectional composite.

2. MODEL GEOMETRY

Consider a unidirectional fibre reinforced composite having a fibre volume fraction V_f and matrix volume fraction V_m such that $V_f + V_m = 1$. The application of axial cyclic loading to the composite is considered to lead to a progressive weakening of the fibres. In addition, interfaces between the fibres and matrix are regarded as being significantly weakened by fatigue loading to the extent that it can be assumed that the fibres and matrix behave independently in regions of the composite that are well away from the uniaxial loading mechanism where clamping effects become important. This assumption means that the composite can be modelled as a parallel bar model, as shown in Fig.1. The fibres in the composite are regarded as acting as a loose bundle forming one bar of the model. The matrix material in the composite is considered as being gathered together to form the other bar of the model which is regarded as homogeneous, i.e. the bar is solid. When a fibre fails the load it carried is shared between the surviving fibres in the bundle and the matrix in such a way that all surviving fibres and matrix experience the same axial strain increment. The fibres are assumed to have the same length so that each surviving fibre has the same stress throughout the progressive failure process. Progressive strength reduction of the fibres due to fatigue loading leads to successive fibre failure until the bundle collapses. It is assumed that bundle collapse corresponds to the catastrophic failure of the composite, i.e. the matrix strength is insufficient to maintain the load when all the fibres have failed.

The composite is subjected to a cyclic uniaxial fatigue load where the R-ratio defined by $R = F_{min}/F$ is fixed; F and $F_{min}\,(= RF)$ denoting the maximum and minimum values of the applied uniform amplitude load cycle. The load range applied to the composite is given by $\Delta F = (1 - R)F$. The number of fatigue cycles executed is denoted by N, where N = 0 corresponds to the instant when the maximum load is first applied. The objective is to develop a parallel bar model of a composite so that it can predict the dependence of composite fatigue life N_f on the fixed maximum load F and load ratio R, and the dependence of the residual strength S(N) of the composite on the number N of fatigue cycles executed from the instant of first loading.

3. BASIC MECHANICS FOR PARALLEL BAR MODEL OF COMPOSITE

The analysis of the parallel bar model shown in Fig.1 will neglect any axial thermal stresses arising from a mismatch of the thermal expansion coefficients of the fibres and the matrix. The area fraction of all fibres in the bundle is denoted by A_b, and that of the matrix by A_m. It follows that

$$V_f = \frac{A_b}{A_b + A_m} , \qquad V_m = \frac{A_m}{A_b + A_m} = 1 - V_f \ . \tag{1}$$

The maximum and minimum loads applied to the fibre bundle after N cycles are denoted by $F_b(N)$ and $RF_b(N)$, and the corresponding applied load range is denoted by $(1-R)F_b(N)$. After N cycles the maximum and minimum stresses during a cycle in each surviving fibre are denoted respectively by $\sigma(N)$ and $R\sigma(N)$, and the corresponding stress range is denoted by $\Delta\sigma(N) = (1-R)\sigma(N)$. The cross-sectional area of each of the fibres in the bundle is denoted by A , and the axial modulus of each fibre is denoted by E_f which is assumed to be independent of the number of cycles that have been executed. The maximum and minimum stresses in the matrix, after N cycles have been executed, are denoted respectively by $\sigma_m(N)$ and $R\sigma_m(N)$ where the corresponding matrix stress range is denoted by $\Delta\sigma_m(N) = (1-R)\sigma_m(N)$. The modulus of the matrix is denoted by E_m which is assumed to be independent of N. A cycle dependence could be included to account for the degradation of the matrix resulting from fatigue loading, but it would be very difficult to develop a model that accounted for the micro-mechanisms of matrix failure.

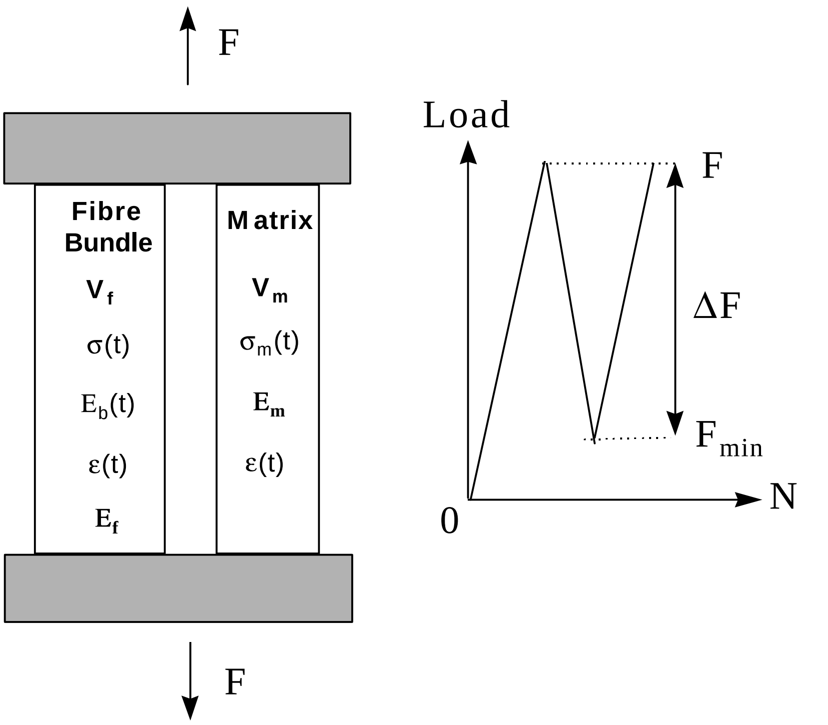

Fig.1 : Two bar model of a unidirectional composite subject to fatigue loading

In Fig.1 the fatigue cycles are represented by a piece-wise linear form that is difficult to achieve in practice where sinusoidal loading is often used. As time dependence is not to be introduced into the model the shape of fatigue cycle assumed is an irrelevant consideration.

After N cycles the maximum axial strain during cycling in all surviving fibres of the bundle and the matrix has the same value that is denoted by $\varepsilon(N)$, and the corresponding minimum value is $R\varepsilon(N)$ such that the strain range is $\Delta\varepsilon(N) = (1-R)\varepsilon(N)$. As thermal expansion mismatch effects are neglected it follows that

$$\varepsilon(N) = \frac{\sigma(N)}{E_f} = \frac{\sigma_m(N)}{E_m} \quad . \tag{2}$$

The balance of forces in the parallel bar model leads to the equilibrium relations

$$F_b(N) + A_m\sigma_m(N) = F \ . \tag{3}$$

The number of surviving fibres in the bundle after N cycles is denoted by $P(N)$ so that the maximum load applied to the bundle after N cycles may be written

$$F_b(N) = P(N)A\sigma(N) \ . \tag{4}$$

On substituting (4) into (3) followed by the elimination of the matrix stress and the strain $\varepsilon(N)$ using (1) and (2) it can be shown that the number of fibres $P(N)$ surviving after N cycles is related to the fibre stress $\sigma(N)$ through the following relation that quantitatively characterises the load sharing that occurs when fibres in the composite fail

$$\left[\frac{P(N)}{P_0} + \alpha\right]\sigma(N) = \frac{F}{A_b} \ , \qquad A_b = P_0 A \ , \qquad P_0 = P(0) \ , \tag{5}$$

where

$$\alpha = \frac{V_m E_m}{V_f E_f} \quad . \tag{6}$$

It is useful to relate the number of fibres surviving in the bundle after N cycles to the effective axial modulus of the bundle $E_b(N)$. The effective stress applied to the bundle is defined by

$$\sigma_b(N) = \frac{F_b(N)}{A_b} \ , \tag{7}$$

and since the axial strain of the bundle and the individual fibres has the value $\varepsilon(N)$

$$\varepsilon(N) = \frac{\sigma(N)}{E_f} = \frac{\sigma_b(N)}{E_b(N)} = \frac{F_b(N)}{A_b E_b(N)} = \frac{P(N)}{P_0}\frac{\sigma(N)}{E_b(N)} \ , \tag{8}$$

where use has been made of (4) and (7). Clearly the effective axial modulus of the fibre bundle is given by

$$E_b(N) = \frac{P(N)}{P_0} E_f \ . \tag{9}$$

The effective stress σ_{app} applied to the composite is defined by

$$\sigma_{app} = \frac{F}{A_b + A_m} = \frac{V_f F}{A_b} , \tag{10}$$

and it can be shown from (1), (5) and (8) that

$$\sigma_{app}(N) = E_c(N)\varepsilon(N) , \quad \text{where} \quad E_c(N) = V_f E_b(N) + V_m E_m , \tag{11}$$

and where $E_c(N)$ is the effective axial modulus of the composite defined by the rule of mixtures, as to be expected.

4. PREDICTING COMPOSITE DEGRADATION

The analysis assumes that the strength of individual fibres is to be determined by a fibre fatigue degradation law, coupled with the assumption that the initial strength of the fibres is statistically distributed according to Weibull weakest link statistics. While one could adopt a fracture mechanics approach to fibre degradation (as was the case when considering environmental degradation [4]), a more general approach will be adopted so that the model has potential for being relevant to other types of fibre degradation mechanism.

4.1 Accounting for fatigue degradation of the fibres

Fibre failure is assumed to be governed by a criterion having the form

$$\sigma^p a = k , \tag{12}$$

where a denotes a statistically distributed damage characteristic of the fibre that measures the amount of fatigue damage in the fibre, i.e. the value of a increases as the fibre becomes progressively fatigued so that its strength decreases. The exponent p and the parameter k are regarded as fibre constants which are always positive.

The fatigue of the fibres is considered to be governed by a damage law of the form

$$\frac{da}{dN} = C \, \Delta\sigma^q a^r , \tag{13}$$

where C, q and r are material constants which are always positive. The parameter a has a similar role to crack length when considering fatigue crack growth.

When fatigue loading is applied to a unidirectional composite the fibre damage characteristic a increases according to the growth law (13) and eventually leads to fibre failure when the failure criterion (12) is satisfied. Thus fibres progressively fail and the load carried by failed fibres is, for the parallel bar model under discussion, transferred to the surviving fibres and matrix using the load sharing rule (5). The stress in each fibre of the system is thus cycle dependent. It is useful to present here the relationship that determines the initial value $X_0(N)$ of the fibre damage characteristic that requires N cycles to grow to the critical value $a_c(N)$ for failure, at which the fibre stress is $\sigma(N)$, under the influence of a cycle dependent fibre stress history $\sigma(n)$; $0 < n < N$. The critical value of the fibre damage characteristic after N cycles is predicted by (12) to be

$$a_c(N) = \frac{k}{\sigma^p(N)} \quad , \tag{14}$$

and it can be shown on integrating (13) between the limits $X_0(N)$ and $a_c(N)$ that

$$X_0(N) = k\left[\sigma^{p(r-1)}(N) + (r-1)\,\lambda\int_0^N \sigma^q(n)\,dn\right]^{\frac{1}{1-r}} \quad , \tag{15}$$

where

$$\lambda = C(1-R)^q\, k^{r-1} \quad . \tag{16}$$

On using (12) it follows from (15) that the initial strength $\sigma_i(N)$ of the fibres, that fail after N cycles when their stress is $\sigma(N)$, is given by

$$\sigma_i(N) = \left[\sigma^{p(r-1)}(N) + (r-1)\,\lambda\int_0^N \sigma^q(n)\,dn\right]^{\frac{1}{p(r-1)}} \quad . \tag{17}$$

The cross-sectional area of the sample of unidirectional composite is assumed to be large enough for there to be a very large number of fibres. It can then be assumed that the bundle of fibres used in the parallel bar model contains every possible fibre strength that can arise in the statistical distribution.

It is assumed that the strength distribution of the fibres is given by the two parameter Weibull distribution [5] so that, for a large bundle of P_0 fibres, the expected number of fibres P surviving when the stress in each fibre is σ is given by

$$P = P_0 \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right] , \tag{18}$$

where m is the Weibull exponent and σ_0 is a scaling parameter that will depend on the length of the composite.

It is useful now to indicate how the values of the parameters p, q and r would be selected if fibre degradation was governed by a fracture mechanics mechanism. In relation (12) defining the fibre failure condition the value of p = 2, and the parameter k is proportional to the square of the fracture toughness of the fibre material. In the damage growth law (13), the value of $q = 2r$ so that the relation may be written in the form of a fatigue crack growth law as follows

$$\frac{da}{dN} = C\left(\Delta\sigma^2\, a\right)^r \quad .$$

The damage growth law then has exactly the same form as the well known Paris fatigue crack growth relationship.

4.2 Prediction of static strength

It is useful to use the parallel bar model to investigate the prediction of the static strength of a unidirectional composite, as in previous work that considered the effects of environmental degradation [4]. When using (18) in conjunction with (5) it is easily shown that the total load carried by the composite, when the stress in the surviving fibres has the value σ, is given by

$$\hat{F} = \hat{\sigma}\left[\alpha + e^{-\hat{\sigma}^m}\right], \qquad (19)$$

where α is defined by (6), and where $\hat{F}$ and $\hat{\sigma}$ are a dimensionless normalised load and stress respectively defined by

$$\hat{F} = \frac{F}{P_0\sigma_0 A}, \qquad \hat{\sigma} = \frac{\sigma}{\sigma_0}. \qquad (20)$$

The static strength of the composite is determined from the maximum value F_{max} of the load that can be carried by the composite. The maximum load occurs when $\hat{F}$ has a local maximum when plotted as a function of $\hat{\sigma}$. The maximum fibre stress σ_{max} satisfies the transcendental equation

$$m\hat{\sigma}_{max}^m = 1 + \alpha\, e^{\hat{\sigma}_{max}^m}. \qquad (21)$$

The corresponding maximum load is then obtained using

$$\frac{F_{max}}{P_0\sigma_0 A} = \hat{F}_{max} = \hat{\sigma}_{max}\left[\alpha + e^{-\hat{\sigma}_{max}^m}\right] = \alpha\,\frac{m\hat{\sigma}_{max}^{m+1}}{m\hat{\sigma}_{max}^m - 1}, \qquad (22)$$

which is consistent with the known result for a loose bundle [2] when the limit $\alpha \to 0$ is taken and identical to the result derived when developing a model for the environmental degradation of a unidirectional composite [4]. The static strength of the composite is simply $V_f\sigma_0\hat{F}_{max}$.

The equation (21) governing the maximum fibre stress does not always have a solution as seen by examining the form of the LHS and RHS of (21). On setting $x = \hat{\sigma}_{max}^m$, the critical conditions defining the limit of solutions to (21) may be written

$$mx = 1 + \alpha e^x, \qquad m = \alpha e^x. \qquad (23)$$

These conditions correspond to the touching of the curves $y = mx - 1$ and $y = \alpha e^x$. It is easily seen that the critical condition occurs when

$$x = \ln\frac{m}{\alpha} = \frac{1+m}{m}. \qquad (24)$$

It is concluded that the equation (21) has a solution only if

$$\alpha < m\,e^{-\left(1+\frac{1}{m}\right)} \quad . \tag{25}$$

If this condition is not satisfied then it is deduced that the fibres progressively fail until there is just one surviving fibre which will then fail, i.e. the bundle does not suddenly collapse. The value of the Weibull modulus m for fibres of interest is usually such that the condition (25) is satisfied so that bundle collapse is always expected in practice.

4.3 Case of fatigue resistant fibres

It is useful here to discuss the case where fibres are highly resistant to fatigue damage. For such cases (e.g. unidirectional carbon fibre composites) the unidirectional composite fails as a result of extensive progressive damage to the matrix and fibre/matrix interfaces leading to increasing stress on the fibres until the bundle of fibres fails statically. The mechanism assumes that the matrix becomes so degraded as a result of fatigue loading that it can no longer support any load. The load carried by the matrix when it is undamaged is progressively transferred to the fibres which acts as a loose bundle causing them to fail if the applied load is large enough.

It is first useful to consider the maximum load that can be supported by the bundle formed from the fibres in the composite without the matrix, thus simulating the effective state of the matrix in the composite following prolonged fatigue loading. It follows from (6) and (22), on setting $E_m = 0$ and $\alpha = 0$, that the maximum load that can be applied to the fibres in the composite without any matrix is given by F_{max}^b where

$$\frac{F_{max}^b}{P_0 \sigma_0 A} = \hat{F}_{max}^b = \hat{\sigma}_{max}^b e^{-\left(\hat{\sigma}_{max}^b\right)^m} .$$

Fig.2 shows the dependence on the parameter α of the normalised maximum load $\hat{F}_{max}^b$, for three values of the Weibull parameter m. It is seen that the normalised maximum load is found to be an almost linear increasing function of α for all values of the Weibull modulus.

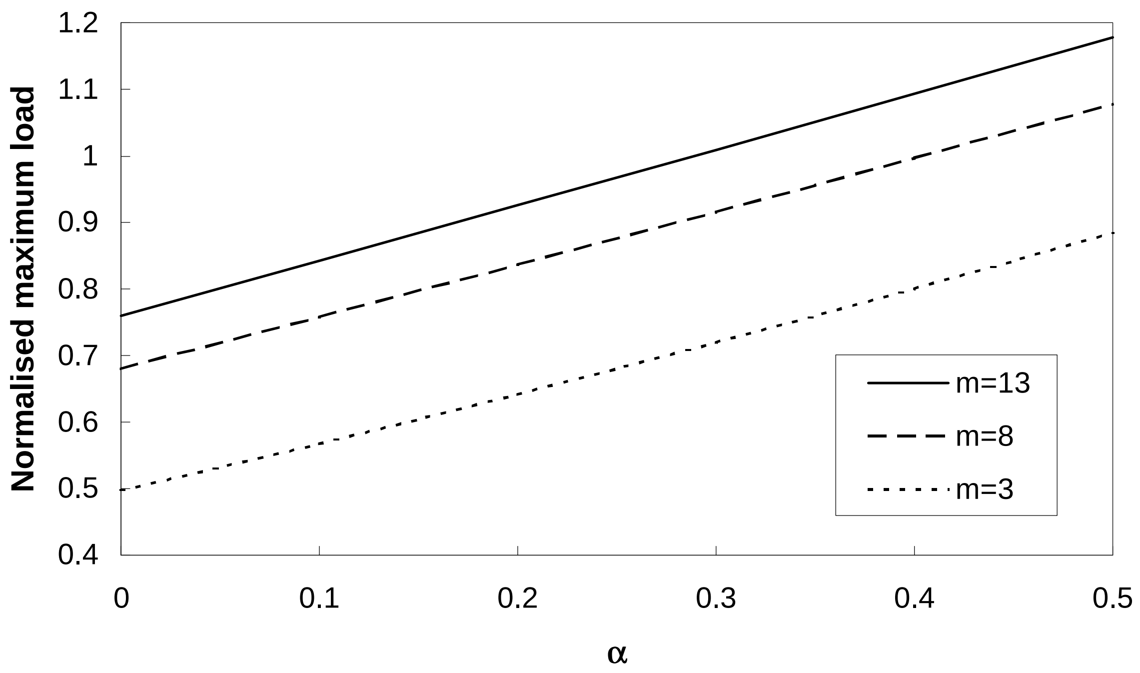

Fig.2 : Dependence of static strength on α and the Weibull exponent.

Based on the values shown in Table 1 for typical fibre and matrix properties of carbon fibre epoxy composites, the parameter α has values which are of the order of 10^{-2} whereas for glass fibre composites the values are of the order of 3×10^{-2}. When m = 8 the percentage difference in the predicted normalised maximum load, between a carbon fibre epoxy composite and a loose fibre bundle with the same number of fibres as the composite, is about 1.2%. For a glass epoxy composite the corresponding percentage difference is about 2.3%. Thus, there is a very small window of applied stress values where progressive matrix failure will lead to composite failure for the case when the fibres are resistant to fatigue degradation. The stress values for S-N data for such materials will be very close to their static strengths. Such behaviour is observed for carbon fibre epoxy composites and it is deduced that carbon fibres are resistant to fatigue damage. Such behaviour is not observed for glass fibre epoxy composites from which it may be deduced that glass fibres are not fatigue resistant. Glass fibres are known to have crack-like defects which can be expected to grow during fatigue loading causing a degradation of fibre properties.

Material	Young's modulus (GPa)	Strain to failure (%)
E-glass fibre	76	4.8
Torayca T300 carbon fibre	230	1.5
Graphil 34-700 carbon fibre	234	1.9
Tenax HTA carbon fibre	238	1.2
913 epoxy	3.39	
922 epoxy	4.05	1.7
924 epoxy	3.8	2.4
196 Crystic Polyester	3.8	1.6
Dion 9100 Vinyl Ester	3.4	5

Table 1 : Fibre and matrix properties

No attempt is made to predict the matrix degradation arising prior to composite failure as the micro-mechanisms involved are exceedingly complex involving matrix cracking, interface debonding, interactions between matrix cracks and debonds, not to mention the local effects of the presence of fibres on micro-crack development during fatigue. The best approach for dealing with matrix degradation that seems tractable is to attempt data modelling where the results of experiments are used to determine the loss of matrix properties as a function of the number of fatigue cycles that have been executed. Such a development is beyond the scope of the current modelling activities.

4.4 Prediction of progressive damage

After N stress cycles the fibres that survive in the composite are those whose initial strengths were greater than $\sigma_i(N)$ defined by (17). It then follows from (18) that the expected number of surviving fibres $P(N)$ after N cycles is given by

$$\frac{P(N)}{P_0} = \hat{P}(N) = \exp\left[-\left(\frac{\sigma_i(N)}{\sigma_0} \right)^m \right] . \qquad (26)$$

On substituting (17) in (26)

$$\left(\ln \frac{1}{\hat{P}(N)} \right)^{\frac{p(r-1)}{m}} = \hat{\sigma}^{p(r-1)}(N) + (r-1)\eta \int_0^N \hat{\sigma}^q(n)dn \ , \tag{27}$$

where

$$\eta = \lambda \sigma_0^{q-p(r-1)} \ , \tag{28}$$

and where use has been made of the definitions (20), which when applied to the load sharing rule (5), lead to

$$\hat{P}(N) = \frac{\hat{F}}{\hat{\sigma}(N)} - \alpha \ . \tag{29}$$

On substituting (29) into (27)

$$\left(\ln \frac{\hat{\sigma}(N)}{\hat{F} - \alpha\hat{\sigma}(N)} \right)^{\frac{p(r-1)}{m}} = \hat{\sigma}^{p(r-1)}(N) + (r-1)\eta \int_0^N \hat{\sigma}^q(n)\, dn \ . \tag{30}$$

On differentiating (30) with respect to N, the following differential equation governing the time dependence of the normalised fibre stress $\hat{\sigma}(N)$ is obtained

$$\left[\frac{1}{m} \frac{\hat{F}}{\hat{F} - \alpha\hat{\sigma}(N)} \left(\ln \frac{\hat{\sigma}(N)}{\hat{F} - \alpha\hat{\sigma}(N)} \right)^{\frac{p(r-1)}{m}-1} - \hat{\sigma}^{p(r-1)}(N) \right] \frac{d\hat{\sigma}(N)}{d(\eta N / p)} = \hat{\sigma}^{q+1}(N) \ . \tag{31}$$

This differential equation is solved by standard numerical techniques subject to the initial condition

$$\hat{\sigma}(0) = s_0 \ , \tag{32}$$

where s_0 is the solution of the transcendental equation

$$\hat{F} = s_0 \left[\alpha + e^{-s_0^m} \right] , \tag{33}$$

corresponding to (19), that must be solved numerically.

4.5 Predicting the failure stress and number of cycles to failure

The structure of the differential equation (31) is such that $d\hat{\sigma} / dN \to \infty$ when $\hat{\sigma}(N) \to \hat{\sigma}_f$ where

$$\frac{1}{m} \frac{\hat{F}}{\hat{F} - \alpha\hat{\sigma}_f} \left(\ln \frac{\hat{\sigma}_f}{\hat{F} - \alpha\hat{\sigma}_f} \right)^{\frac{p(r-1)}{m}-1} = \hat{\sigma}_f^{p(r-1)} \ . \tag{34}$$

The stress $\hat{\sigma}_f$ in the surviving fibres when the composite fails can thus be determined using numerical methods without having to solve the differential equation (31). It should be noted that when $\hat{F} = \hat{F}_{max}$ the solution of (34) is given by $\hat{\sigma}_f = \hat{\sigma}_{max}$ where $\hat{F}_{max}$ and $\hat{\sigma}_{max}$ are given by (21) and (22).

The transcendental equation (34), that usually must be solved numerically, involves the dimensionless loading parameter $\hat{F}$ in a complicated way. It is useful to unravel the dependence on this parameter by using the load sharing rule (29) to express (34) in terms of $\hat{P}_f$ the fraction of surviving fibres just before failure. The load sharing rule is first written at the point of failure in the form

$$\hat{\sigma}_f = \frac{\hat{F}}{\hat{P}_f + \alpha} , \tag{35}$$

and then the relation (34) reduces to the form

$$\hat{F}^{p(r-1)} = \frac{1}{m} \frac{\left(\hat{P}_f + \alpha\right)^{p(r-1)+1}}{\hat{P}_f} \left(\ln \frac{1}{\hat{P}_f} \right)^{\frac{p(r-1)}{m} - 1} . \tag{36}$$

Having solved (36) numerically to find the value of $\hat{P}_f$, the corresponding value $\hat{\sigma}_f$ for the fibre stress at the point of failure is given using (35). The number of cycles to failure N_f can be predicted only by solving the differential equation (31) in the normalised stress range $s_0 \le \hat{\sigma}(N) \le \hat{\sigma}_f$.

4.6 Predicting residual strength

A key requirement concerning the effects of fatigue loading on composite degradation is the prediction of the cycle dependence of the residual strength of a composite. This has already been considered for the case of a bundle of loose fibres [3], and for a unidirectional composite subject to environmental degradation [4]. The objective now is to show how the residual strength of a unidirectional composite with weak interfaces can be predicted. After N stress cycles have elapsed where the maximum and minimum loads are fixed, the load is instantaneously increased until the composite fails catastrophically. Just before the load is suddenly increased the stress in the surviving fibres has the value $\sigma(N)$ and at any stage during the subsequent instantaneous load increase the value of the stress in the fibres is denoted by s . When the fibre stress has the value s the critical value of the fibre damage characteristic has the following value specified by (12)

$$a_c^* = \frac{k}{s^p} . \tag{37}$$

It is necessary to calculate the original value X^* of the critical fibre damage characteristic using (12) and (13). It is easily shown that

$$\left(X^*\right)^{1-r} = \left(a_c^*\right)^{1-r} + C(1-R)^q (r-1) \int_0^N \sigma^q(n)\, dn . \tag{38}$$

On using (12) the initial strength of the fibres, that are critical after N fatigue cycles have been executed and the fibre stress has the value s during subsequent instantaneous loading, is denoted by s_i and is obtained from the following relation derived from (38)

$$\hat{s}_i^{p(r-1)} = \hat{s}^{p(r-1)} + (r-1)\eta \int_0^N \hat{\sigma}^q(n)\, dn \ , \tag{39}$$

where use has been made of (16) and (28), and where

$$\hat{s}_i = \frac{s_i}{\sigma_0} \quad , \qquad \hat{s} = \frac{s}{\sigma_0} \quad . \tag{40}$$

On using (30) the relation (39) may be written in the form

$$\hat{s}_i^{p(r-1)} = \hat{s}^{p(r-1)} - K(N) \ , \tag{41}$$

where

$$K(N) = \hat{\sigma}^{p(r-1)}(N) - \left(\ln \frac{\hat{\sigma}(N)}{\hat{F} - \alpha\hat{\sigma}(N)} \right)^{\frac{p(r-1)}{m}} \quad . \tag{42}$$

The load applied to the composite F_s, when the fibre stress has the value s, is obtained from (20) and (29) so that

$$\frac{F_s}{P_0\sigma_0 A} = \hat{F}_s = \hat{s}\left[\alpha + \hat{P}_s \right] , \tag{43}$$

where $\hat{F}_s$ is the normalised applied load and where $\hat{P}_s$ is the normalised number of surviving fibres when the load on the composite is such that the fibre stress has the value s. It follows from (18) that

$$\hat{P}_s = e^{-\hat{s}_i^m} \quad . \tag{44}$$

On substituting in (43) the following expression is derived for the normalised load applied to the composite during a residual strength test

$$\hat{F}_s = \hat{s}\left[\alpha + e^{-\hat{s}_i^m} \right] . \tag{45}$$

The residual strength S(N) of the composite after N stress cycles is the maximum value of F_s when s is varied, or alternatively the maximum value of $\hat{F}_s$ when $\hat{s}$ is varied. Noting that K(N) is independent of $\hat{s}$, the maximum value of $\hat{F}_s$ occurs when $\hat{s}_i = x(N)$ which satisfies the transcendental equation which must be solved numerically to determine the value of x(N)

$$1 + \alpha e^{x^m(N)} = m\, x^m(N) \left(1 + \frac{K(N)}{x^{p(r-1)}(N)} \right) . \tag{46}$$

On using (41) the stress $\sigma_{max}(N)$ in the surviving fibres just before the composite fails during a residual strength test is obtained by substituting the solution x(N) of (46) into the following relation

$$\frac{\sigma_{max}(N)}{\sigma_0} = \left[x^{p(r-1)}(N) + K(N) \right]^{\frac{1}{p(r-1)}} = \hat{\sigma}_{max}(N) . \tag{47}$$

It then follows from (41) and (45) that the residual strength of the composite S(t) is obtained using

$$\frac{S(N)}{\sigma_0} = \hat{\sigma}_{max}(N) \left[\alpha + e^{-x^m(N)} \right] = \hat{S}(N) . \tag{48}$$

When N = 0 it can be shown using (39) that $\hat{s}_i = \hat{s}$ so that K(0) = 0 in which case the transcendental equation (46) reduces to the form (21) which needs to be solved when calculating the static strength of the composite.

5. PRELIMINARY RESULTS

In order to assess the properties of the fatigue model some preliminary predictions have been made to illustrate the principal characteristics. There are seven parameters that need to be specified in order to obtain preliminary predictions that illustrate the characteristics of the fatigue model. The first is the Weibull exponent m characterising the strength distribution of the fibres before fatigue loading commences. This parameter, which often has values in the range 4 - 8, appears in the relation (18) defining the expected number of fibre failures for a given fibre stress. Preliminary predictions will assume that m = 8. The value of m is usually obtained from single fibre strength tests. The second parameter p appearing in the fibre failure criterion is taken to have the value p = 2 so that the failure criterion is then consistent with fracture mechanics. The third parameter is the exponent q appearing in the fibre degradation law (13). This parameter is taken to have the value q = 12 which is seen below to lead to the situation where there is only a very gradual reduction in the residual strength during most of the test. Near to failure the residual strength decreases rapidly as fatigue loading proceeds. The fourth parameter is the exponent r that also appears in the fibre degradation law. The value r = 6 is selected so that the fibre degradation law is corresponds in form to the well known form of fatigue growth law based on the stress range intensity factor. The fifth parameter α, defined by (6), takes account of the properties of the fibre and matrix, and also of the volume fraction.

Predictions will be made for the value α = 0.03 which is a value representative of glass fibre composites. When $\alpha = 0$ the composite is being modelled in exactly the same way as a bundle of loose fibres. The sixth parameter that needs to be specified is the level of loading applied to the composite. The model assumes that the ratio F/F_{max} is given where F is the axial maximum load applied to the composite during fatigue, and where F_{max} is the maximum load that can be supported by the composite before exposure to fatigue loading. The value of the ratio F/F_{max} always lies in the range $0 \rightarrow 1$. The seventh and final parameter is the fatigue ratio R which is taken as zero for the predictions indicating that the minimum applied stress during a fatigue cycle is zero.

The Euler/Richardson solution technique [6] is used to solve the differential equation (31) where the parameter $\eta N / p$ is regarded as the unknown function of $\hat{\sigma}$ of the differential equation. The initial condition is specified by (32) and (33) and the range $s_0 \leq \hat{\sigma} \leq \hat{\sigma}_f$ is subdivided into 1000 equal intervals when solving the differential equation. The upper bound $\hat{\sigma}_f$ is determined by the relations (35) and (36).

Fig.3 shows the dependence of the normalised stress σ / σ_0 in the fibres as a function of the number of fatigue cycles N, where N_f is the number of cycles to failure. The results are for the case when F/F_{max} = 0.6 and for two values α = 0 and α = 0.03. It is seen that progressive fibre failure resulting from fatigue loading leads to an increase in the stress of the surviving fibres. The rate of increase of fibre stress is seen to tend to infinity as the failure point is approached. For the parameters chosen for the prediction, the effect of the matrix on fibre stress profile is very small. This means that the composite is behaving as a loose bundle of fibres where the matrix is ignored.

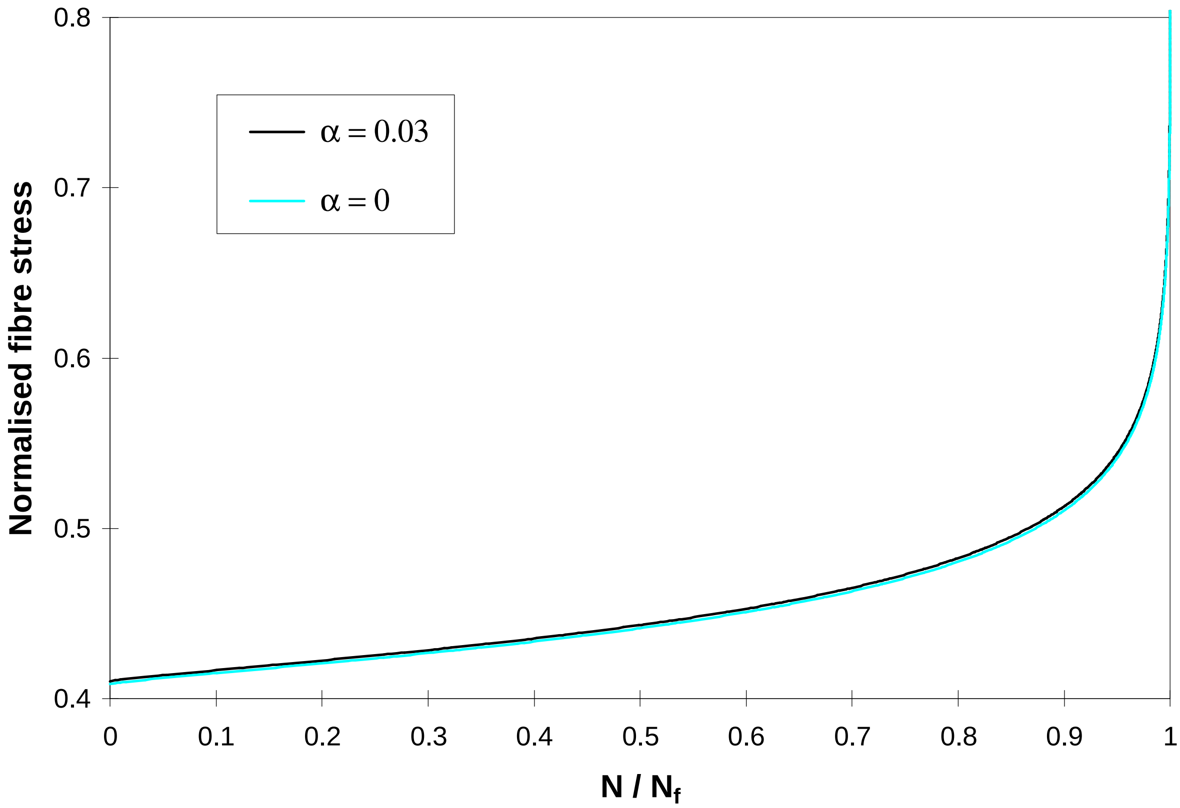

Fig.3 : Prediction of the fibre stress as a function of cycles when a = 0, 0.03 for the case F/F_{max} = 0, p = 2 , q = 12, r = 6 and R = 0.

Fig.4 shows the results of predicting both the S-N curve (shown as a thick black line), and the residual strength curves as a function of the number of fatigue cycles that have been executed at the point where the residual strength was estimated. The abscissa is a logarithmic scale which indicates only the decades over which the results are obtained. Exactly the same shape of curve would be obtained if the results were plotted as a function of $\log_{10}N$. The residual strength curves are seen to depend on the value of F/F_{max} chosen when carrying out the N fatigue cycle before the residual strength test is carried out. It should be noted that the residual strength curves all terminate at the S-N curve, as they should since at the last point the additional stress needed to cause failure in a residual strength test is zero, i.e. the residual strength is the value given by the S-N curve.

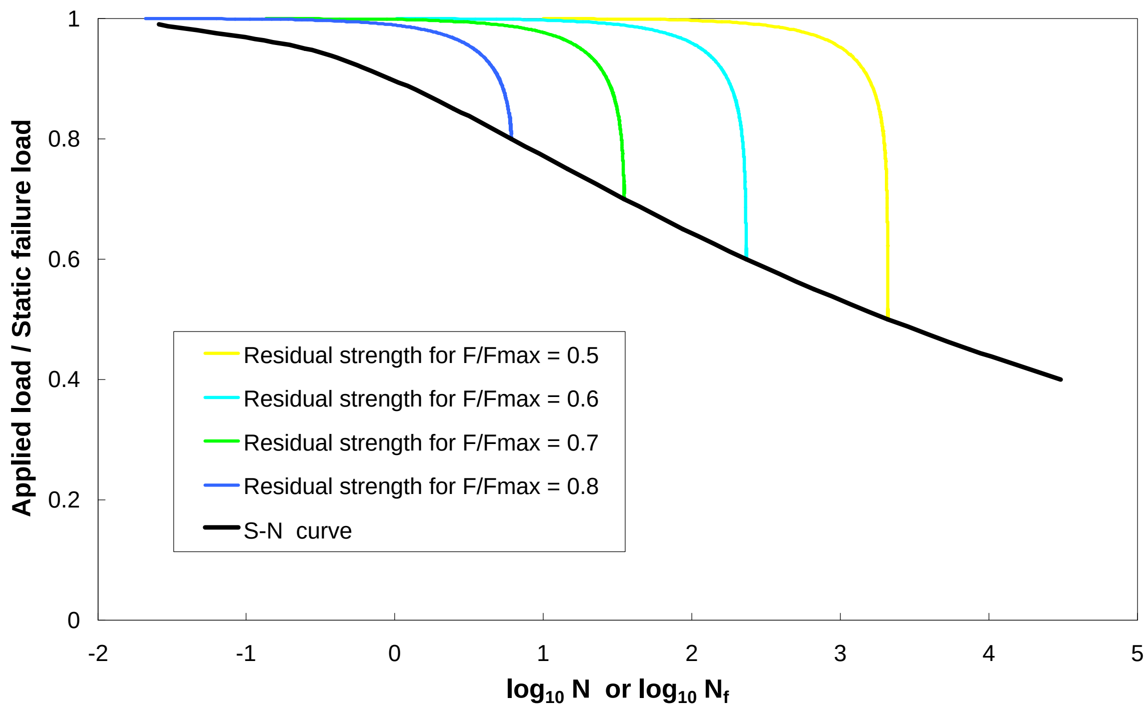

Fig.4 : Prediction of S-N curve and residual strength for the case p = 2 , q = 12, r = 6, a = 0.03 and R = 0.

Results, corresponding to those in Fig.4 where α = 0.03 typical of glass fibre composites, have also been generated for the case α = 0 which is assuming that the matrix effect is negligible. The S-N curve differs very little indeed from that shown in Fig. . The differences in the predicted normalised lives is between 0.6% and 0.8%. The percentage difference will be even smaller for carbon fibre composites. This means that when fatigue life data and residual strength data are plotted as shown in Fig.4, such that the ordinate is the ratio F/F_{max} which is identical to the applied stress divided by the static strength, the results are virtually independent of the properties of the matrix. In other words the same data could be obtained by testing a bundle of loose fibres to predict the S-N curve and the residual strength curve. This results is emphasising the fact that the fatigue model presented assumes that fatigue performance is dominated by the behaviour of the fibres. A similar conclusion was reached when modelling the environmental degradation of a unidirectional composite [4].

6. DISCUSSION & CONCLUSIONS

From the work carried out in the project, it is clear that the development of a universal validated model of fatigue damage in a unidirectional composite is an exceedingly difficult task, if not impossible. It appears that models will have to be tailored to specific situations, and that a specific model may not be able to predict its own range of validity. The validation of models through comparison of predictions with experimental results is also very difficult as the conditions relating to the failure of testpieces do not usually correspond to the definition of failure assumed in the model. In this report a specific model has been developed that is based on the concept of damage development in fibres. For such a model the predicted results can be expected to be fibre dominated, and this has been found to be the case from preliminary predictions.

One conclusion to be drawn from the preliminary results presented in this report, is that the cycle dependence of the axial properties of a unidirectional fibre reinforced composite subject to fatigue loading can be predicted using a parallel bar model of the composite where interface bonding is neglected. The model can also be used to predict the cycle dependence of the residual strength of the composite. It has been shown that, when the residual strength of the composite is divided by its static strength, the resulting strength ratio is virtually independent of the matrix properties. It is concluded that the strength ratio for a unidirectional composite, where fibres are prone to fatigue damage, is predictable from the static strength of the composite, and the cycle dependence of the residual strength of a bundle of loose fibres. A corollary to this is that the fatigue performance of unidirectional composites might be estimated experimentally from an examination of the fatigue behaviour of a bundle of loose fibres. There are experimental difficulties that would be encountered (e.g. gripping) but these are no more severe than those encountered when fatigue testing unidirectional composites using more conventional methods where it is required that the fatigue failure is not associated with the testpiece gripping mechanism, and that failure does not arise from an unrepresentatively large defect in the composite.

The modelling and validation difficulties encountered are a characteristic of the fact that a unidirectional composite is being considered where extremely complex damage modes contribute to the fatigue failure of the composite. The next stage of the project is concerned with the fatigue performance of cross-ply laminates. For this case matrix damage is more readily modelled as it usually first occurs in the form of micro-cracks in the 90^o plies which then grow in size forming larger cracks which traverse the thickness of ply. Such cracks will then grow in the transverse direction until the edge of the laminate is reached i.e. the ply cracks are fully developed. A complication is the occurrence of delamination initiating from laminate edges or from fully developed transverse cracks. Fatigue damage may also occur in the 0^o plies and the model developed in this report will be used for composites where fibre degradation due to fatigue is an important damage mechanism.

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The theory in the report is used in software application BUNDLES\FATIGUE.