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Approximate Estimation of the Effects of Asymmetry on the Testing of Tensile Specimens

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APPROXIMATE ESTIMATION OF THE EFFECTS OF ASYMMETRY ON THE TESTING OF TENSILE SPECIMENS

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SUMMARY

An analysis is made of the plane stress problem representing the tensile testing of rectangular orthotropic material specimens. The effects of various geometric non-symmetries are considered and some approximate polynomial solutions are obtained and tabulated. Comparisons of the theory with specific finite element calculations show good general agreement.

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1 INTRODUCTION

Tensile testing techniques play a central role in the measurement of mechanical properties of materials, both in terms of stiffness and failure properties. Methods for the measurement of induced strain include extensometers, moiré techniques and the use of strain gauges. In the latter case, these are attached to the specimen at one or more positions in order to determine the effects of the loading. A proposed ASTM and ISO standard describes an alignment procedure based on strain gauges. Requirements on the position of these gauges is clearly part of the specification of a testing standard. Difficulties arise when the specimen is not mounted symmetrically within the machine; this changes the load conditions significantly with subsequent distortion of the strain field. Information on these effects is necessary in order to determine how much asymmetry can be tolerated in a useful test. The purpose of the work to be described here is to estimate the effects of various asymmetries on the stress, strain and displacement distributions over the specimen. It is hoped that this information will assist both those involved in the preparation of testing standards and those performing the experimental testing.

A relevant aspect of the problem concerns the local effects of the grips on the specimen. A constraint requiring all displacements and rotations to be zero is extremely difficult or impossible to obtain in practice. In many cases, the specimen tends to pull out of the clamp due to Poisson contractions in the thickness direction. Various grip designs are used to minimise this effect. These include geometric and hydraulic wedge action grips that apply increasing through-thickness loads as longitudinal load is increased. However the problem can persist. In addition, early local material failures due to stress concentrations can occur in corner regions, resulting in further local relaxation of constraints. That these effects are local follows from Saint-Venant's Principle [1]. Essentially, this states that a change in the distribution of an end load, without change in resultant, alters the stress significantly only near that end. Exactly how near is 'near' is not defined, although it is accepted that this implies a distance of the same order of magnitude as the dimensions of the loading area of the body. In the present context, a knowledge of the extent of the edge region is desirable in order that strain gauges can be located outside it. The point must be made that approximate methods of analysis of the whole specimen of the kind used here will <u>not</u> yield information on the extent of edge effects. Some knowledge of this effect can be obtained by finite element analysis of specific cases, but two difficulties arise. The first has been mentioned earlier, namely that detailed information on the form of the edge constraint provided by the grip is not available. Finite element analysis requires a full definition of the boundary conditions in terms of the type (eg one or more displacements or rotations) and on its distribution (ie applied to the whole edge or only part of it). Several combinations of restraints are clearly possible. The other difficulty concerns the treatment of possible singularities that can arise near a corner. The use of a high mesh density is necessary to resolve these regions thus making the calculations expensive. In addition it is likely that the shape functions used by the element do not properly represent the form of the singularity.

The present work, while not addressing the edge effect problem, is aimed at producing approximate formulae for the stress, strain and displacement distributions for various geometric asymmetries. In this way, the role of the material and geometric parameters can be identified.

The next section of the report describes the problems for treatment and Section 3 gives details of the mathematical analysis. Subsequent sections set out the approximate solutions and give some comparisons with finite element results.

2 LOADING CONFIGURATIONS

In this section we list the configurations to be analysed in later sections. We consider first the case in which the sample is incorrectly mounted as shown in Figure 1. Here the specimen centre line is offset from the grip centre line. On loading, an axial displacement is imposed on the specimen but the grips are allowed to rotate; this produces the situation shown in Figure 1(a) in which the specimen becomes curved. We shall call this Case 1 (Offset Placement with End Rotation).

A case related to the above is shown in Figure 2. Here the specimen is mounted correctly before loading, but during the extensional loading, the grips are made to rotate symmetrically as shown in Figure 2(a). This is described by a known gradient of the centre line at the end of the specimen.

We consider next the configuration indicated in Figure 3. Here the sample is positioned correctly before loading. During loading however, due to machine deficiencies, some lateral displacement of the grips takes place in addition to the main longitudinal displacement. This produces the result shown schematically in Figure 3(a). We point out here that for purposes of illustration, the figures show magnified effects; in reality the asymmetries may be quite small.

Finally we consider the case shown in Figure 4, in which the specimen is mounted skew to the grips before loading. The amount of skew is given by the angle θ as is also the relative displacement u_o of the grips after loading.

3 GOVERNING EQUATIONS

For purposes of the analysis, we shall assume that the specimens are of orthotropic material with the 1 direction aligned longitudinally with the x axis, and the 2 or transverse direction aligned with the y axis. We assume that a state of plane stress is applicable during loading and ignore the effect of any through-thickness compression applied by the action of the grip design. In the absence of body forces, the equations of equilibrium [1] are:-

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} &= 0 \end{aligned}\right\} \qquad (3.1)$$

where σ_x, σ_y denote the normal stresses and τ_{xy} is the shear stress. We introduce a stress function ϕ that satisfies these equations identically by defining

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} , \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} , \quad \tau_{xy} = - \frac{\partial^2 \phi}{\partial x \partial y} . \qquad (3.2)$$

The orthotropic stress-strain relationships [2] are

$$\varepsilon_x = \frac{\sigma_x}{E_1} - \nu_{21}\frac{\sigma_y}{E_2} , \tag{3.3}$$

$$\varepsilon_y = \frac{\sigma_y}{E_2} - \nu_{12}\frac{\sigma_x}{E_1} , \tag{3.4}$$

and

$$\gamma_{xy} = \tau_{xy}/G_{12} . \tag{3.5}$$

We have also the following relationship between the material constants

$$\nu_{12} E_2 = \nu_{21} E_1 . \tag{3.6}$$

In terms of the displacements, the strains are

$$\varepsilon_x = \frac{\partial u}{\partial x} , \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3.7}$$

where u and v are the horizontal (x) and vertical (y) displacements respectively. In terms of the stress function ϕ the condition of compatibility becomes [1], [2]

$$\frac{1}{E_2}\frac{\partial^4\phi}{\partial x^4} + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{1}{E_1}\frac{\partial^4\phi}{\partial y^4} = 0 . \tag{3.8}$$

For an isotropic material $E_1 = E_2 = E$, $\nu_{12} = \nu_{21} = \nu$, $G_{12} = E/2(1+\nu)$ and the above equation reduces to the standard biharmonic equation. Various solutions to (3.8) are required subject to boundary conditions in terms of applied forces and displacements. Certain boundary conditions are common to all the cases to be considered; these state that the longitudinal edges are stress-free. If c is the half-width of a specimen, then $y = \pm c$ denotes these edges; the common conditions are therefore

$$\sigma_y = \tau_{xy} = 0 \quad \text{on} \quad y = \pm c . \tag{3.9}$$

The conditions at the other edges are provided by the clamping and loading configurations; these will be detailed later as the individual cases are analysed.

4 ANALYSIS OF THE LOADING CONFIGURATIONS

4.1 Case 1. Offset Placement with End Rotation

This case is symmetric about x = 0; the co-ordinate system, dimensions and applied loading displacements are shown in Figure 1(b). We shall assume that known horizontal displacements of u_o are applied at $y = y_o$ on AC and BD as shown. In addition, we shall assume that the edge BD is free to turn about the point E, so that there is no applied moment about E. Clearly there are no shear forces applied at AC and BD; because of this and (3.9) we shall assume that the shear stress τ_{xy}, is zero throughout. We shall also assume that σ_y is identically zero throughout the specimen as no normal forces are applied at $y = \pm c$ due to (3.9). These conditions are satisfied by a stress function of the form

$$\phi = \frac{\alpha y^2}{2} + \frac{\beta y^3}{6} \qquad (4.1.1)$$

where α and β are constants to be determined by the conditions on $x = \pm \ell$. This also satisfies the compatibility equation (3.8).

If t denotes the thickness of the specimen, the absence of any turning moment about E is expressed by

$$\int_{-c}^{c} (y-y_o)\, \sigma_x dy = 0 \ . \qquad (4.1.2)$$

Substitution of (4.1.1) into (3.2) gives

$$\sigma_x = \alpha + \beta y \ , \qquad \sigma_y = 0 \ , \qquad \tau_{xy} = 0 \ .$$

Condition (4.1.2) then yields

$$\frac{\beta c^3}{3} = \alpha y_o c$$

or

$$\alpha/\beta = c^2/3y_o \quad . \tag{4.1.3}$$

From (3.7) and (3.3) we have

$$\frac{\partial u}{\partial x} = \frac{\sigma_x}{E_1} = \frac{1}{E_1}(\alpha + \beta y) \quad .$$

Integration with respect to x gives

$$u = \frac{x}{E_1}(\alpha + \beta y) \tag{4.1.4}$$

as u = 0 when x = 0 due to symmetry. However, the displacement at (ℓ, y_o) is u_o,

$$u_o = \frac{\ell}{E_1}(\alpha + \beta y_o) \quad . \tag{4.1.5}$$

From (4.1.3) and (4.1.5) we obtain

$$\alpha = \frac{E_1 u_o}{\ell} \frac{c^2}{c^2 + 3y_o^2}, \quad \beta = \frac{E_1 u_o}{\ell} \cdot \frac{3y_o}{c^2 + 3y_o^2} \quad .$$

Again from (3.7) and (3.4)

$$\frac{\partial v}{\partial y} = \frac{-\nu_{12}}{E_1}\sigma_x = \frac{-\nu_{12}}{E_1}(\alpha + \beta y) \quad .$$

Integration with respect to y gives

$$v = \frac{-\nu_{12}}{E_1}\left(\alpha y + \frac{\beta y^2}{2}\right) + g(x)$$

where g(x) is an arbitrary function of x. Substitution of this and (4.1.4) into the third part of (3.7) and remembering that τ and therefore γ_{xy} is zero for this case gives

$$\frac{x\beta}{E_1} + g'(x) = 0$$

or

$$g(x) = \frac{-\beta x^2}{2E_1} + \text{constant} .$$

Choosing $v = 0$ at $x = \ell$, $y = y_o$ (ie the point E is constrained to move horizontally) we have finally

$$v = \frac{\nu_{12}}{E_1}\left[\alpha(y_o - y) + \frac{\beta}{2}(y_o^2 - y^2)\right] + \frac{\beta}{2E_1}(\ell^2 - x^2) . \quad (4.1.6)$$

Examination of (4.1.6) indicates that the centre line vertical displacement is parabolic in x; the gradient of v is given by

$$\frac{\partial v}{\partial x} = \frac{-\beta x}{E_1} . \quad (4.1.7)$$

This gives a rotation of $\beta\ell/E_1$ for the grips at the end of the specimen. An expression can be obtained for the total force exerted by the grips; this is given by

$$F = \int_{-c}^{c} t\sigma_x \, dy = t\int_{-c}^{c} (\alpha + \beta y)dy = 2\alpha ct \quad (4.1.8)$$

where t is the thickness of the specimen. Figure 1(b) summarises the main results of this sub-section.

4.2 Case 2. Symmetric placement with enforced end rotation

Again this case is symmetric about $x = 0$; we use the same co-ordinate system as before, shown this time in Figure 2(b). Now the point of application of the end displacement u_o is at the centre of the edge BD. The enforced rotation of this edge is θ. We choose again

$$\varphi = \frac{\alpha y^2}{2} + \frac{\beta y^3}{6} \quad (4.2.1)$$

and obtain as for the previous case

$$\sigma_x = \alpha + \beta y, \quad \sigma_y = \tau_{xy} = 0 \quad .$$

The equation for the horizontal displacement u is again given by (4.1.4). Putting $u = u_o$ at $x = \ell$, $y = 0$ in (4.1.4) gives $u_o = \alpha\ell/E_1$, from which

$$\alpha = E_1 u_o/\ell \quad .$$

Putting y_o to zero in (4.1.6), we obtain for the vertical displacement

$$v = \frac{-\nu_{12}}{E_1}\left[\alpha y + \frac{\beta y^2}{2}\right] + \frac{\beta}{2E_1}(\ell^2 - x^2) \qquad (4.2.2)$$

where we have assumed $v = 0$ at $x = \ell$, $y = 0$. From this equation

$$\frac{\partial v}{\partial x} = \frac{-\beta x}{E_1} \quad .$$

However, the inclination of the specimen is given to be θ at $x = \ell$, $y = 0$. So

$$\beta = -\frac{E_1 \tan\theta}{\ell} \quad . \qquad (4.2.3)$$

Again, the horizontal force exerted by the grips is

$$F = 2\alpha ct \quad .$$

The turning moment required to produce the rotation is given by

$$M = t\int_{-c}^{c} \sigma_x y\,dy = \frac{2\beta tc^3}{3} \quad .$$

Figure 2(b) shows the main results for this case.

4.3 Case 3. Lateral displacement

The co-ordinate system for analysing this case is shown in Figure 3(b). Note that here the total length of specimen is ℓ and width 2c. As indicated in Figure 3(a), the edge BD is constrained to undertake displacements in both directions relative to the centre of AC. We shall assume that the displacement of the centre of BD is given as (u_o , v_o). We shall require further boundary conditions representing constraints on rotation of the specimen. If the performance of the grips were perfect then all points on BD would have the same displacement. As mentioned before this state is nearly impossible to obtain in practice; we shall instead impose $\partial u/\partial y = 0$ on the mid-points ($y = 0$) of BD and AC as suggested in [3]. This implies some loss of gripping efficiency and resulting relaxation of the constraint. A further relaxation would be described by the use instead of the condition $\partial v/\partial x = 0$ at these points. This would require a horizontal gradient on the centre line but would allow variation of u in the y direction. This may model somewhat gripping failure or pull-out near the corners of the specimen. We shall give details of this solution but consider it of less physical importance.

This case clearly involves the application of traction forces over the edges BD and AC and so some variation of the shear stress is necessary. We choose the stress function to be of the form

$$\varphi = \frac{\alpha y^2}{2} + \frac{\beta y^3}{6} + \delta xy\left(1 - \frac{y^2}{3c^2}\right) \qquad (4.3.1)$$

which satisfies (3.8).

Using (3.2), the choice (4.3.1) gives $\sigma_y = 0$ and $\tau_{xy} = -\delta(1 - y^2/c^2)$ thus satisfying the stress free conditions on $y = \pm c$. It follows that

$$\sigma_x = \alpha + \beta y - 2\delta \frac{xy}{c^2} . \qquad (4.3.2)$$

Using (3.7) and (3.3), we have as before

$$\frac{\partial u}{\partial x} = \frac{\sigma_x}{E_1}$$

and so

$$u = \frac{1}{E_1}\left(\alpha x + \beta yx - \frac{\delta yx^2}{c^2}\right) + f(y) \ .$$

Similarly

$$\frac{\partial v}{\partial y} = \frac{-\nu_{12}}{E_1}\sigma_x$$

and

$$v = \frac{-\nu_{12}}{E_1}\left(\alpha y + \frac{\beta y^2}{2} - \frac{\delta xy^2}{c^2}\right) + g(x) \ .$$

Substitution of these expressions for u and v into the third part of (3.7) gives

$$\frac{1}{E_1}\left(\beta x - \frac{\delta x^2}{c^2}\right) + f'(y) - \frac{\nu_{12}}{E_1}\left(\frac{-\delta y^2}{c^2}\right) + g'(x) = \frac{-\delta}{G_{12}}\left(1 - \frac{y^2}{c^2}\right) \ .$$

For this relation to hold for all x and y we must have

$$\frac{1}{E_1}\left(\beta x - \frac{\delta x^2}{c^2}\right) + g'(x) = a$$

and

$$\left(\frac{\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\frac{\delta y^2}{c^2} + f'(y) = b$$

where a and b are constants satisfying $a + b = -\delta/G_{12}$. These expressions can be integrated to give f(y) and g(x); we have finally

$$u = \frac{1}{E_1}\left(\alpha x + \beta xy - \frac{\delta yx^2}{c^2}\right) + \left(\frac{1}{G_{12}} - \frac{\nu_{12}}{E_1}\right)\frac{\delta y^3}{3c^2} + by + e$$

$$v = \frac{\nu_{12}}{E_1}\left(\alpha y + \frac{\beta y^2}{2} - \frac{\delta xy^2}{c^2}\right) - \frac{1}{E_1}\left(\frac{\beta x^2}{2} - \frac{\delta x^3}{3c^2}\right) + ax + d \ .$$

We choose to measure all displacements from the origin; this gives $e = d = 0$. Using $a + b = -\delta/G_{12}$ to eliminate b we have

$$u = \frac{1}{E_1}\left(\alpha x + \beta xy - \frac{\delta yx^2}{c^2}\right) + \left(\frac{1}{G_{12}} - \frac{\nu_{12}}{E_1}\right)\frac{\delta y^3}{3c^2} - \left(a + \frac{\delta}{G_{12}}\right)y \qquad (4.3.3)$$

$$v = \frac{-\nu_{12}}{E_1}\left(\alpha y + \frac{\beta y^2}{2} - \frac{\delta xy^2}{c^2}\right) - \frac{1}{E_1}\left(\frac{\beta x^2}{2} - \frac{\delta x^3}{3c^2}\right) + ax \ . \qquad (4.3.4)$$

It remains to determine the four constants α, β, δ and a. Two conditions are supplied by the known displacements u_o, v_o at $x = \ell$, $y = 0$. Two further conditions are supplied by *either*

$$\left(\frac{\partial u}{\partial y}\right) = 0 \qquad \text{at } x=0 \text{ and } \ell \text{ on } y=0 \qquad (4.3.5)$$

or

$$\left(\frac{\partial v}{\partial x}\right) = 0 \qquad \text{at } x=0 \text{ and } \ell \text{ on } y=0 \ . \qquad (4.3.6)$$

Two quantities of interest which can be derived from the solution are the total normal force produced by the grips:-

$$F = \int_{-c}^{c} t\sigma_x dy$$

and the total shear traction

$$T = \int_{-c}^{c} t\tau_{xy}\, dy \quad .$$

Substitution of (4.3.2) and subsequent evaluation give

$$F = 2\alpha ct$$

and

$$T = \frac{-4\delta tc}{3} \quad .$$

Using first the conditions (4.3.5) with the displacement conditions yields

$$\alpha = \frac{E_1 u_o}{\ell}, \quad \beta = \frac{-6G_{12}E_1 v_o}{D}, \quad \delta = \frac{-6c^2 G_{12} E_1 v_o}{\ell D}, \quad a = \frac{6c^2 E_1 v_o}{\ell D} \qquad (4.3.7)$$

where $\qquad D = 6c^2 E_1 + G_{12}\ell^2 \quad .$

The use of conditions of (4.3.6) instead of (4.3.5) gives

$$\alpha = \frac{E_1 u_o}{\ell} \quad \beta = \frac{-6E_1 v_o}{\ell^2}, \quad \delta = \frac{-6c^2 E_1 v_o}{\ell^3}, \quad a = 0 \quad .$$

Figure 3(b) summarises the main results for lateral displacement.

4.4 Case 4. Skew placement

This configuration is shown schematically in Figure 4. Prior to loading, the specimen is mounted in a skewed fashion at an inclination of θ to the line of centres of the grips. During loading, we shall assume the right hand grip moves horizontally a distance u_o with respect to the left hand grip. This configuration differs from those considered so far in that, strictly speaking, this problem requires the analysis of a slightly skewed plate (θ assumed small) instead of a rectangular one. We shall, however, approximate this case by considering the specimen to be rectangular but subjected to skew edge conditions. In this way we shall be ignoring the effects of small triangular regions near the grips. The geometry for this case is shown in Figure 4(b); the length of the specimen is ℓ and the width is 2c. The horizontal applied displacement u_o in the skewed configuration can be resolved into $u_o \cos\theta$ in the x direction and $u_o \sin\theta$ in the negative y direction as shown in the diagram. This is in contrast to Case 3 in which u_o and v_o were defined independently. The rotational constraint of most interest for Case 3 is given by (4.3.5) in which $\partial u/\partial y$ is zero at both ends of the specimen. This requires modification in the present case to allow for the skewed geometry. It is shown in the Appendix that the required condition is

$$\cos^2\theta \frac{\partial u}{\partial y} + \sin\theta\cos\theta \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \sin^2\theta \frac{\partial v}{\partial x} = 0 \text{ at } x=0 \text{ and } \ell \text{ on } y=0. \qquad (4.4.1)$$

which clearly reverts to (4.3.5) for $\theta = 0$.

We see from the above that this configuration can be derived from Case 3 by putting $u_o = u_o \cos\theta$, $v_o = -u_o \sin\theta$ in the formulae for that case and using (4.4.1) instead of (4.3.5). The solution is then

$$\alpha = \frac{E_1 \cos\theta\, u_o}{\ell}$$

$$\beta = \frac{6\,B \sin\theta\, G_{12}\, E_1\, u_o}{D}$$

$$\delta = \frac{6\,B \sin\theta\, c^2 G_{12} E_1\, u_o}{\ell\, D}$$

$$a = \frac{[(1+\nu_{12})\, G_{12}\,\ell^2 - 6c^2 E_1] \sin\theta \cos^2\theta\, u_o}{\ell\, D}$$

where $B = 1 + \cos^2\theta\,(1+\nu_{12})$, $D = G_{12}\,\ell^2 + 6\cos^2\theta\, c^2 E_1$.
Formulae for this case are summarised in figure 4(b).

5 RESULTS AND DISCUSSION

Some comparisons of the results of the theory described in the previous sections were made with finite element calculations. The material was assumed to be isotropic with elastic properties

$$E_1 = E_2 = 139 \text{ GPa}, \nu = 0.3 .$$

We consider first results for Case 1. Dimensions for the specimen were assumed to be

$$\ell = 0.15 \text{ m}, \quad c = 0.0075 \text{ m}, \quad t = 0.001 \text{ m}$$

whereas the loading parameters were

$$y_o = 0.00375 \text{ m}, \quad u_o = 0.001 \text{ m}.$$

The reader is reminded that the full specimen is of length 2ℓ and width 2c. The finite element calculation was performed using PAFEC with ten elements in the longitudinal direction and four in the transverse direction. The element used was a simple four-noded rectangular element which allowed for in-plane loads only. Table 1 shows strains and displacements for the theory and finite element calculations for three points at $x = 0$ (specimen centre) and three points at $x = \ell/2$. Examination of the table shows that the displacements agree to better than 20% everywhere but results for the strains show less consistency. Where strains are large, agreement can be as good as 5%, but the agreement is poorer at $y = -c$ (the lower concave edge).

Table 2 shows comparison of finite element results with those obtained from Section 4.3 for Case 3 (Lateral Displacement). The elastic constants and specimen dimensions are as before, although ℓ now denotes the full specimen length. For this case $u_o = 0.001$ m and $v_o = -0.0005$ m. The finite element calculation was again performed with 4 x 10 elements. Results are shown for $x = \ell/2$ and $x = \ell$ (at the end of the specimen). The table shows good agreement between FE and theory displacement results, mostly within 5%. Agreement is generally good also for the strain figures except for the ε_y values at $x = \ell$. However, we would not expect good agreement at this position due to end loading effects. Generally, the table shows surprisingly good agreement considering the relative simple nature of the approximating polynomials.

Summaries of the formulae and solutions for the various cases are given with the corresponding schematic drawings, Figures 1(b) to 4(b). These figures thus represent data sheets for the different cases.

Figures 5, 6 and 7 show contour plots of ε_x, u and v respectively for a case 4 example (skew placement). The material and geometric data are identical to that used earlier, together with u_o = 0.001 m and skew angle θ = 0.1 radians (5.7 degrees). Figure 5 shows a cross shaped region situated at the centre of the specimen over which the strain is approximately constant. Figure 6 indicates an almost linear variation of u in the longitudinal direction with very little dependence on y for this case. Finally Figure 7 shows linear variations of v over only the central region of the specimen; the quadratic and cubic terms clearly dominate nearer the ends.

6 APPLICATION

As an example of how the analyses of Section 4 may be used, we consider the configuration corresponding to Case 3. The summary sheet of Figure 3(b) gives the following approximate expression for the strain in the x direction as

$$\varepsilon_x = \frac{1}{E_1}\left(\alpha+\beta y-\frac{\delta xy}{c^2}\right),$$

with corresponding definitions for α, β and δ in terms of the material properties and loading displacements u_o and v_o. Putting $x = \ell/2$ in this expression gives the variation of ε_x with y across the specimen half way between the grips. From this the difference in ε_x across the width of the specimen is

$$\varepsilon_x(\ell/2,c)-\varepsilon_x(\ell/2,-c) = \frac{1}{E_1}\left(\alpha+\beta c-\frac{\delta\ell}{2c}\right)-\frac{1}{E_1}\left(\alpha-\beta c+\frac{\delta\ell}{2c}\right)$$

$$= \frac{1}{E_1}\left(2\beta c-\frac{\delta\ell}{c}\right).$$

Substitution for β and δ from summary sheet 3(b) yields the following expression for the difference in strain:-

$$\frac{-6cG_{12}v_o}{(6c^2E_1+G_{12}\ell^2)}$$

which is proportional to v_o, the imposed transverse displacement. Thus strain gauges at these positions would indicate a difference in ε_x which can be related to the asymmetry parameter v_o. Similarly, difference in strain and other quantities at other positions can be estimated and related to the loading conditions. Conversely, if a test standard requires that changes in strain between certain positions should be less than a certain specified amount, the above expression can be used to determine the maximum allowable value of the asymmetry parameter v_o. Clearly, the results for the other loading configurations can be used in a similar fashion.

7 CONCLUDING REMARKS

Previous sections have described how simple polynomials can be used to approximate solutions to various plane stress problems associated with tensile testing. Some comparisons with finite element calculations indicate that the solutions are adequate to give first order approximations to the stress, strain and displacement distributions over the specimen. However, these solutions do not take account of localised effects and should not be applied in the region of the end grips. In order to use the results presented here the operator should first attempt to identify the configuration which corresponds most closely to the conditions imposed by the apparatus and operating conditions. This will not always be a straightforward task, and it may be necessary to consider more than one configuration.

It is of interest to note how the effect of orthotropy enters into the solutions. For Cases 1 and 2, the non-dependence of the formulae on E_2 and G_{12} indicates that the results are identical to those that would be obtained for an isotropic material. In contrast, orthotropic properties play a role in the solutions for Cases 3 and 4. This is explained by the presence of shearing effects applied at the grip positions.

The configurations treated here are not meant to be exhaustive; other restraints and conditions are possible, although the uncertainties in the practical aspects of the gripping conditions may well give rise to difficulties in the modelling and lead to results of doubtful value. The simple nature of the models located in this report enables them to be readily incorporated into PC programs concerned with tensile testing.

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Co-ords (x,y)	ε_x THEORY	ε_x FE	ε_y THEORY	ε_y FE	ux10^3 THEORY	ux10^3 FE	vx10^3 THEORY	vx10^3 FE
(0,c)	0.0095	0.0090	-0.0028	-0.0023	0	0	8.56	7.01
(0,0)	0.0038	0.0042	-0.0011	-0.00092	0	0	8.58	7.03
(0,-c)	-0.0019	-0.00034	0.00057	-0.00025	0	0	8.58	7.03
(ℓ/2,c)	0.0095	0.0090	-0.0029	-0.0023	0.71	0.68	6.42	5.26
(ℓ/2,0)	0.0038	0.0044	-0.0011	-0.0017	0.28	0.33	6.43	5.27
(ℓ/2,-c)	-0.0019	-0.00034	0.00057	-0.00025	-0.14	-0.026	6.43	5.27

TABLE 1 Comparison with finite element calculation for Case 1.

$E_1 = E_2 = 139$ GPa, $\nu = 0.3$, $G_{12} = 53.46$ GPa
$\ell = 0.15$ m, $c = 0.0075$ m, $y_o = 0.00375$ m, $t = 0.001$ m, $u_o = 0.001$ m

Co-ords (x,y)	ε_x THEORY	ε_x FE	ε_y THEORY	ε_y FE	ux10^3 THEORY	ux10^3 FE	vx10^3 THEORY	vx10^3 FE
(ℓ/2,c)	0.0067	0.0063	-0.0020	-0.0019	0.536	0.554	-0.265	-0.264
(ℓ/2,0)	0.0067	0.0063	-0.0020	-0.0019	0.500	0.529	-0.25	-0.25
(ℓ/2,-c)	0.0067	0.0063	-0.0020	-0.0019	0.464	0.504	-0.235	-0.236
(ℓ,c)	0.0057	0.0062	-0.0017	-5x10^{-5}	1	1.025	-0.51	-0.50
(ℓ,0)	0.0067	0.0060	-0.002	5x10^{-7}	1	1	-0.50	-0.50
(ℓ,-c)	0.0076	0.0062	-0.0023	-5x10^{-5}	1	0.975	-0.48	-0.50

TABLE 2 Comparison with finite element calculation for Case 3.

$E_1 = E_2 = 139$ GPa, $\nu = 0.3$, $G_{12} = 53.46$ GPa

$\ell = 0.15$ m, $c = 0.0075$ m, $t = 0.001$ m, $u_o = 0.001$ m, $v_o = -0.0005$ m

APPENDIX

Modification of Boundary Condition to allow for Rotation of Axes.

Let OX,OY be co-ordinate axes such that OX lies along the centre line of the grips shown in Figure 4. Let Ox,Oy be axes rotated through θ from OX,OY as shown in Figure I. These will correspond to x and y axes used in the analysis of Case 4. Thus we have

$$\begin{aligned} x &= X\cos\theta + Y\sin\theta \\ y &= -X\sin\theta + Y\cos\theta \ . \end{aligned} \qquad \text{(A.1)}$$

If U denotes the displacement of a particle in the X direction and u and v are the corresponding horizontal and vertical displacements in the Ox, Oy axes as shown in Figure II, then we have similarly

$$U = u\cos\theta - v\sin\theta \ . \qquad \text{(A.2)}$$

The requirement is to rotate the boundary condition

$$\frac{\partial U}{\partial Y} = 0 \qquad \text{(A.3)}$$

into the Ox,Oy system. We make a change of independent variables

$$\frac{\partial}{\partial Y} = \frac{\partial x}{\partial y}\frac{\partial}{\partial x} + \frac{\partial y}{\partial Y}\frac{\partial}{\partial y} \ .$$

Using (A.1) this becomes

$$\frac{\partial}{\partial Y} = \sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y} \ .$$

On using (A.2) we have

$$\frac{\partial U}{\partial Y} = \sin\theta\frac{\partial}{\partial x}(u\cos\theta - v\sin\theta) + \cos\theta\frac{\partial}{\partial y}(u\cos\theta - v\sin\theta)$$

or

$$\frac{\partial U}{\partial Y} = \cos^2\theta \frac{\partial u}{\partial y} + \sin\theta\cos\theta\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \sin^2\theta \frac{\partial v}{\partial x} .$$

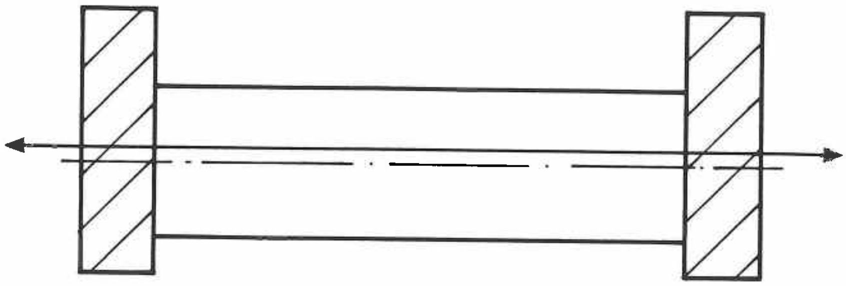


Figure 1 Offset placement before loading

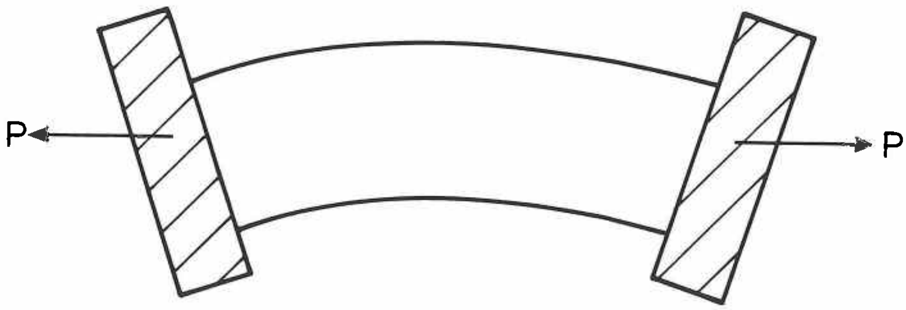


Figure 1(a) Offset after loading with ends free to rotate

Case 1 Offset Placement with End Rotation

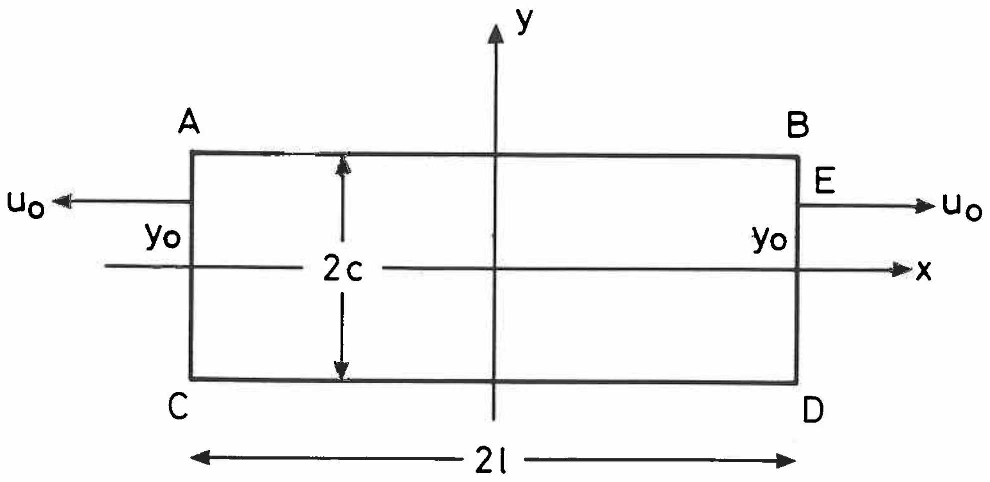


Figure 1(b) Geometry for Case 1
Offset Placement with End Rotation

SOLUTION SUMMARY (Section 4.1)

$$\sigma_x = \alpha + \beta y, \; \sigma_y = 0, \; \tau_{xy} = 0$$

$$\varepsilon_x = \frac{1}{E_1}\sigma_x, \; \varepsilon_y = -\frac{\nu_{12}}{E_1}\sigma_x$$

$$u = \frac{x}{E_1}(\alpha + \beta y), \; v = \frac{\nu_{12}}{E_1}\left[\alpha(y_o - y) + \frac{\beta}{2}(y_o^2 - y^2)\right] + \frac{\beta}{2E_1}(l^2 - x^2)$$

$$\alpha = \frac{E_1 u_o}{l} \cdot \frac{c^2}{c^2 + 3y_o^2}, \quad \beta = \frac{3y_o\alpha}{c^2}$$

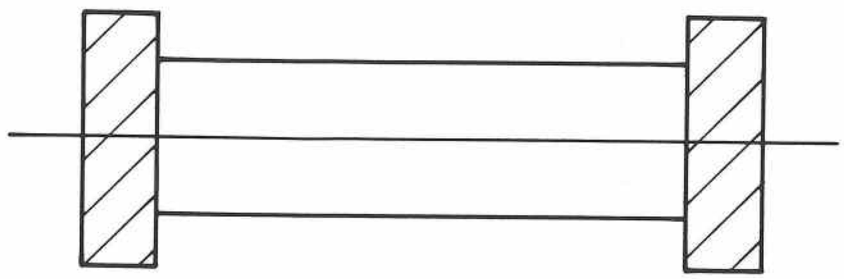


Figure 2 Alignment before loading

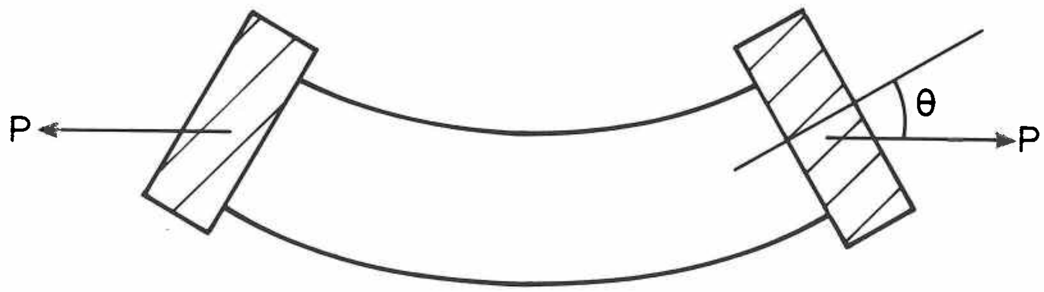


Figure 2(a) Enforced end rotation during loading

Case 2 Symmetric Placement with Enforced End Rotation

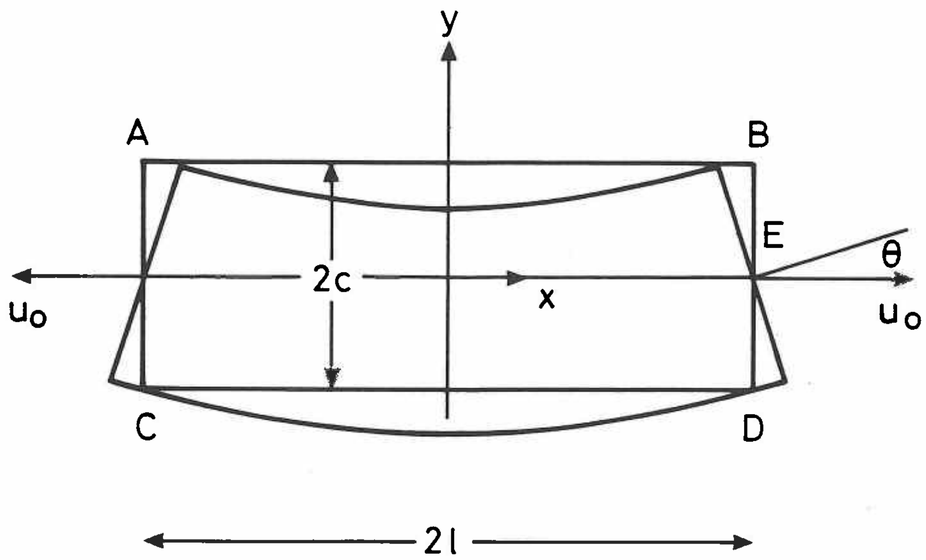


Figure 2(b) Geometry for Case 2
Symmetric Placement with
Enforced End Rotation

SOLUTION SUMMARY (Section 4.2)

$$\sigma_x = \alpha + \beta y,\ \sigma_y = 0,\ \tau_{xy} = 0$$

$$\varepsilon_x = \frac{1}{E_1}\sigma_x \qquad \varepsilon_y = -\frac{\nu_{12}\sigma_x}{E_1}$$

$$u = \frac{x}{E_1}(\alpha + \beta y),\ v = -\frac{\nu_{12}}{E_1}\left(\alpha y + \frac{\beta y^2}{2}\right) + \frac{\beta}{2E_1}(l^2 - x^2)$$

$$\alpha = \frac{E_1 u_o}{l},\ \beta = -\frac{E_1 \tan\theta}{l}$$

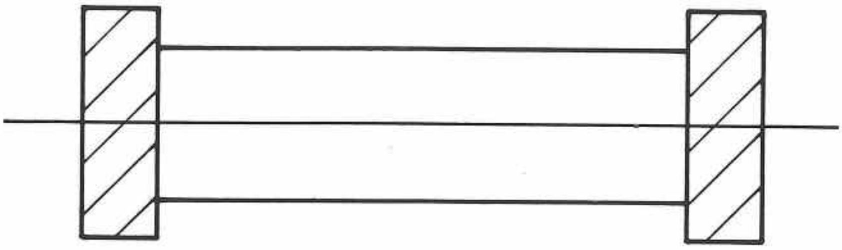


Figure 3 Before loading

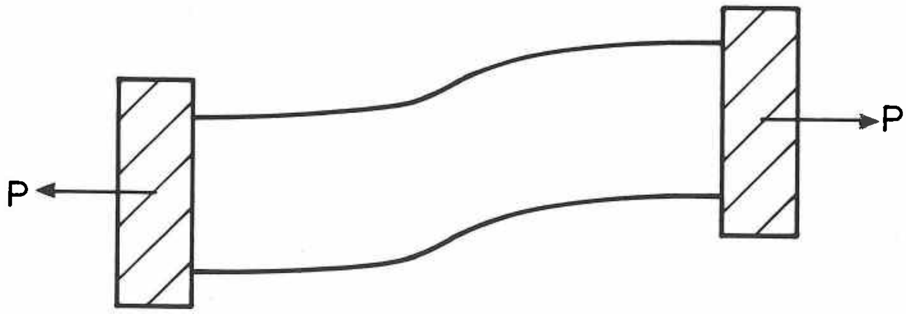


Figure 3(a) Lateral displacement of grips after loading

Case 3 Lateral Displacement of Grips

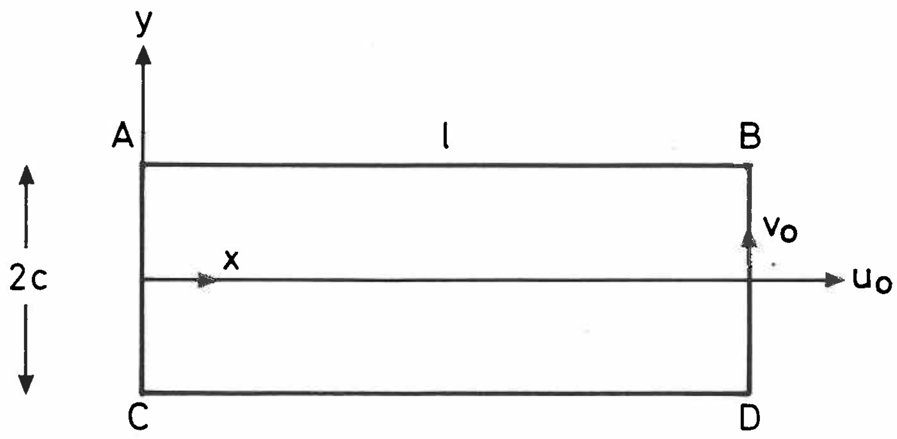


Figure 3(b) Geometry for Case 3
Lateral Displacement

SOLUTION SUMMARY (Section 4.3)

$$\sigma_x = \alpha + \beta y - \frac{2\delta xy}{c^2}, \quad \sigma_y = 0, \quad \tau_{xy} = -\delta(1 - y^2/c^2)$$

$$\varepsilon_x = \frac{1}{E_1}\sigma_x, \quad \varepsilon_y = -\frac{\nu_{12}}{E_1}\sigma_x$$

$$u = \frac{x}{E_1}\left(\alpha + \beta y - \frac{\delta xy}{c^2}\right) + \left(\frac{1}{G_{12}} - \frac{\nu_{12}}{E_1}\right)\frac{\delta y^3}{3c^2} - \left(a + \frac{\delta}{G_{12}}\right)y$$

$$v = \frac{-\nu_{12}}{E_1}\left(\alpha y + \frac{\beta y^2}{2} - \frac{\delta x y^2}{c^2}\right) - \frac{1}{E_1}\left(\frac{\beta x^2}{2} - \frac{\delta x^3}{3c^2}\right) + a x$$

For $\partial u/\partial y = 0$ at $x = 0, l, y = 0$

$$\alpha = \frac{E_1 u_o}{l}, \quad \beta = -\frac{6G_{12}E_1 v_o}{D}, \quad \delta = \frac{\beta c^2}{l}, \quad a = -\frac{\beta c^2}{lG_{12}}$$

$$D = 6c^2 E_1 + G_{12} l^2$$

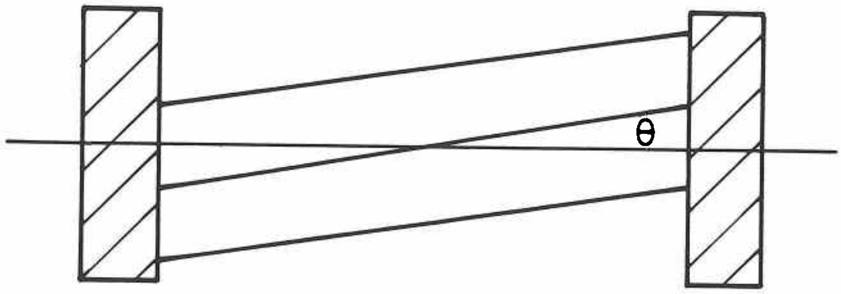


Figure 4 Before loading

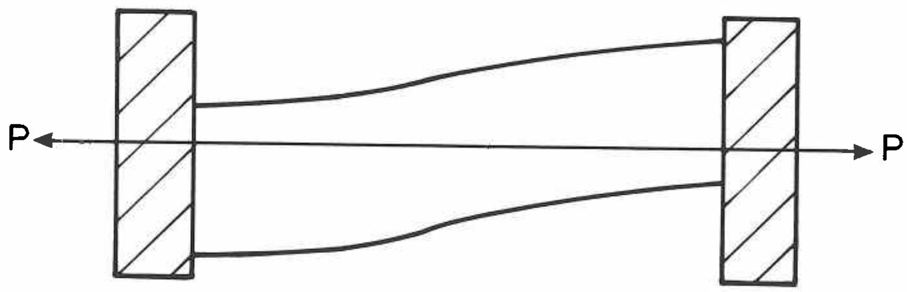


Figure 4(a) After loading

Case 4 Skew Placement

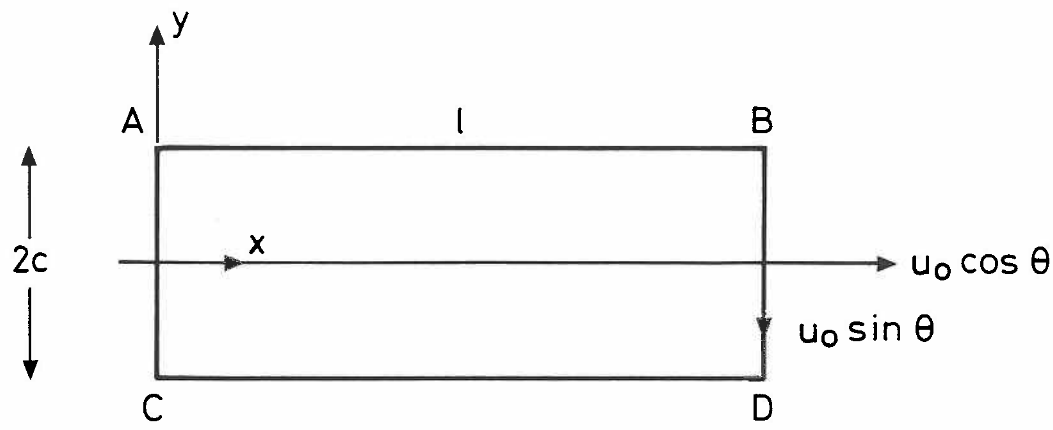


Figure 4(b) Geometry for Case 4
Skew Placement

SOLUTION SUMMARY (Section 4.4)

$$\sigma_x = \alpha + \beta y - \frac{2\delta xy}{c^2}, \; \sigma_y = 0, \tau_{xy} = -\delta\,(1 - y^2/c^2)$$

$$\varepsilon_x = \frac{1}{E_1} \cdot \sigma_x \,, \; \varepsilon_y = -\frac{\nu_{12}}{E_1}\, \sigma_x$$

$$u = \frac{x}{E_1}\left(\alpha + \beta y - \frac{\delta xy}{c^2}\right) + \left(\frac{1}{G_{12}} - \frac{\nu_{12}}{E_1}\right)\frac{\delta y^3}{3c^2} - \left(a + \frac{\delta}{G_{12}}\right) y$$

$$v = -\frac{\nu_{12}}{E_1}\left(\alpha y + \frac{\beta y^2}{2} - \frac{\delta xy^2}{c^2}\right) - \frac{1}{E_1}\left(\frac{\beta x^2}{2} - \frac{\delta x^3}{3c^2}\right) + a x$$

$$\alpha = \frac{E_1 \cos\theta\, u_o}{l}, \; \beta = \frac{6B \sin\theta\; G_{12} E_1 u_o}{D}, \; \delta = \frac{c^2\beta}{l}$$

$$a = \frac{\left[(1+\nu_{12})\, G_{12}\, l^2 - 6c^2 E_1\right] \sin\theta\; \cos^2\theta\, u_o}{lD}$$

$$B = 1 + \cos^2\theta\,(1+\nu_{12}) \qquad D = G_{12}\, l^2 + 6\cos^2\theta\, c^2 E_1$$

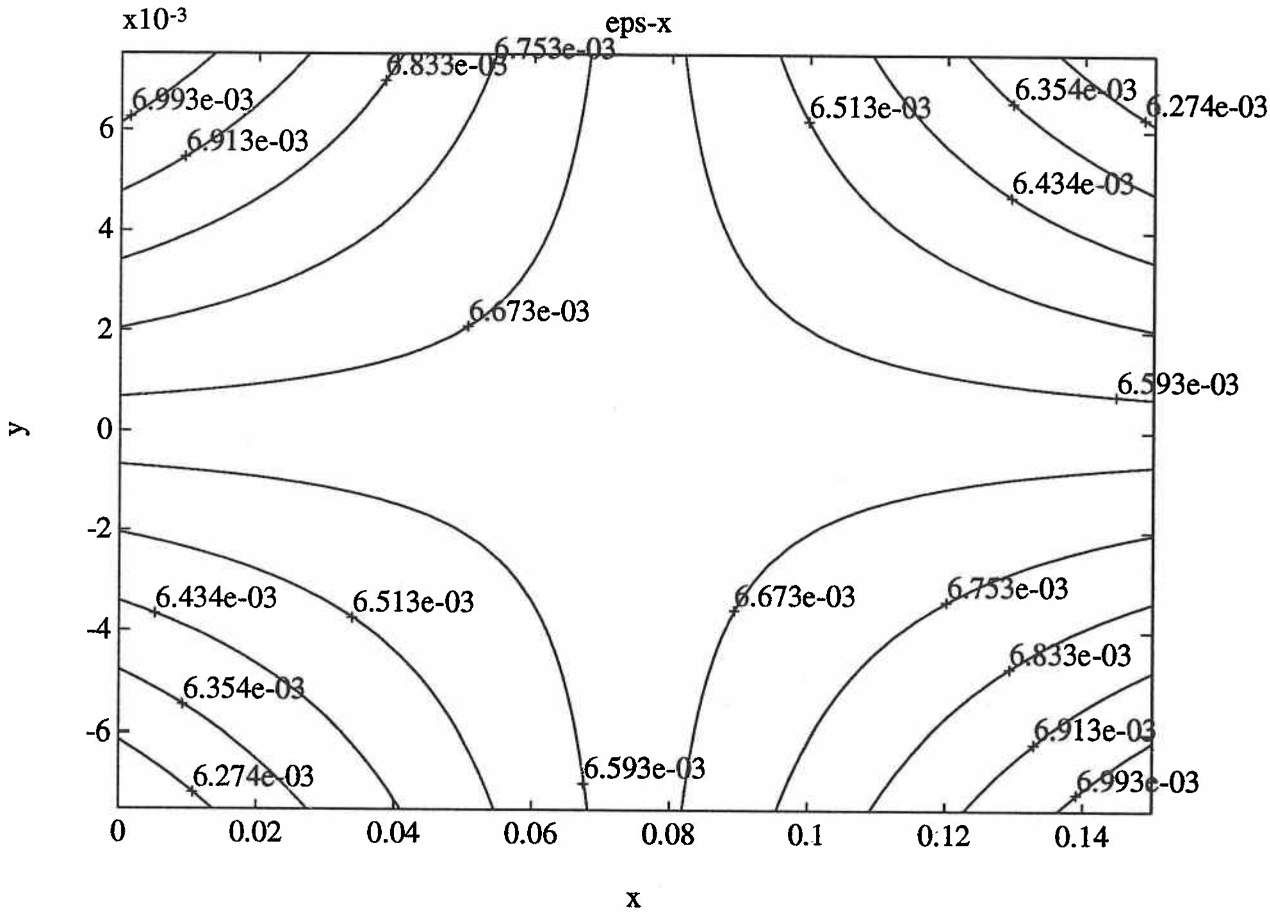
Figures 5, 6 and 7

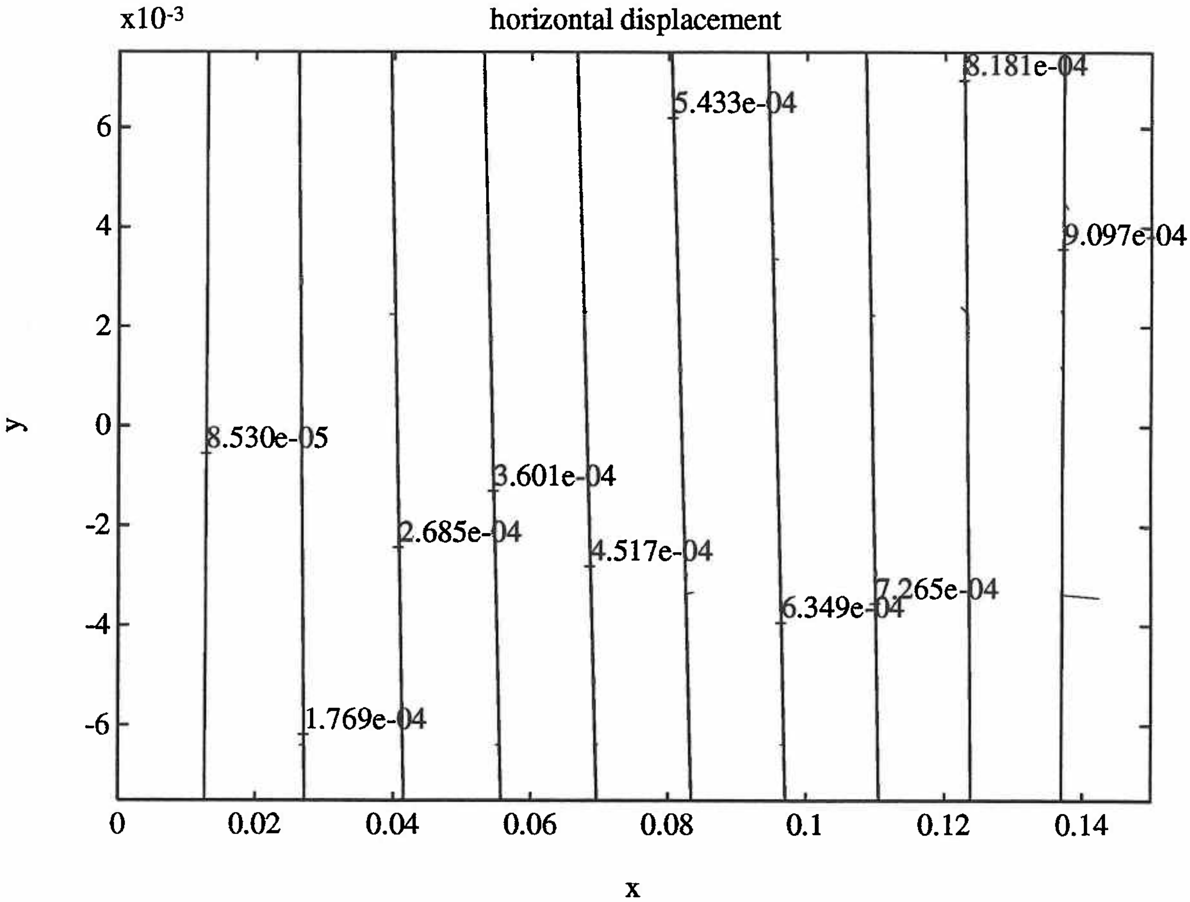
Skew placement

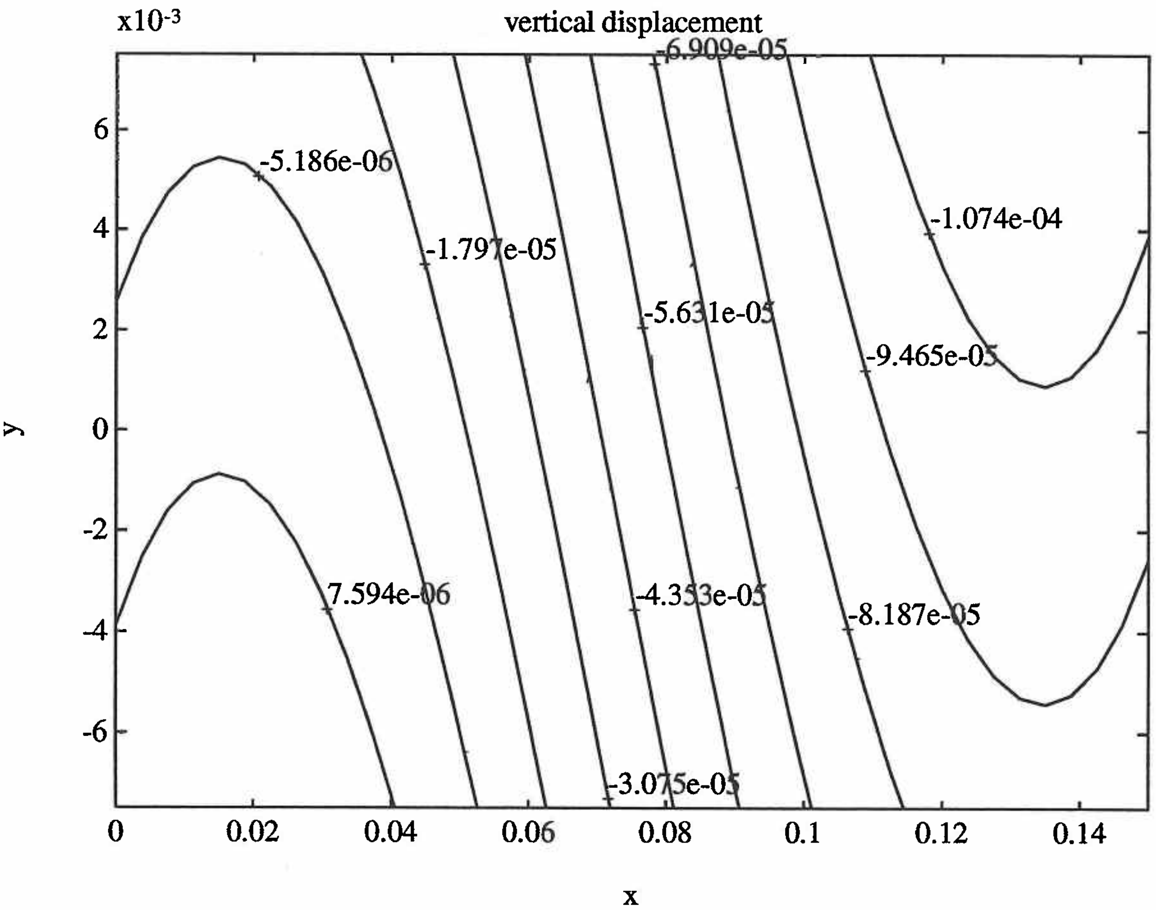
Longitudinal strain and displacements

l = 0.15m, c = 7.5000e-03m, theta = 0.1

E1 = 1.3900e+11Pa, nu12 = 0.3, G12 = 5.3462e+10Pa, u0 = 1.0000e-03m




x10-3
horizontal displacement
8.181e-04
5.433e-04
9.097e-04
8.530e-05
3.601e-04
2.685e-04
4.517e-04
7.265e-04
6.349e-04
1.769e-04
6
4
2
0
-2
-4
-6
y
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
x


x10-3
vertical displacement
-6.909e-05
-5.186e-06
-1.074e-04
-1.797e-05
-5.631e-05
-9.465e-05
7.594e-06
-4.353e-05
-8.187e-05
-3.075e-05
6
4
2
0
-2
-4
-6
y
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
x

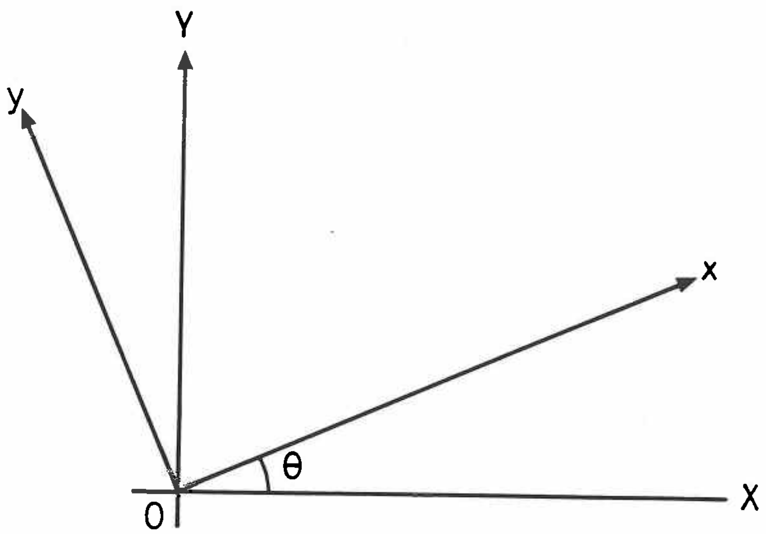


Figure I Rotation of Axes

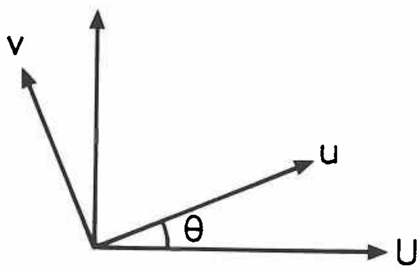


Figure II Displacement Diagram