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MODELLING FUTURE IMPACT IN RESPONSE TO CHANGES IN PUBLIC R&D INVESTMENT INTO NPL

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We are the UK's National Metrology Institute (NMI), a world-leading centre of excellence that provides cutting-edge measurement science, engineering and technology that underpins prosperity and quality of life in the UK.

Abstract

This report presents a framework for forecasting impact from R&D investment in the form of number of firms supported by the National Physical Laboratory (NPL), offering insights into the long-term impact of funding decisions. The report adapts the principle of maximum entropy to derive the probability distribution of marginal value gained by a firm through an instance of support from NPL and quantifies its relationship with NPL's accumulated stock of knowledge built through sustained R&D investment. Three scenarios – an increase, a decrease, and a one-year omission in R&D spend – are analysed to illustrate the implications for firm engagement.

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Contents

Executive Summary

This report presents a comprehensive framework for quantifying the relationship between the National Physical Laboratory's (NPL) research and development (R&D) expenditure and the number of firms that access its support over six-year periods. The approach is designed to align with the six-year analysis model that is standard in IEA (Impact Evaluation and Assessment) reports at NPL (see 1.2), to ensure consistency with established evaluation methodologies. The analysis draws on economic theory and statistical mechanics to model the decision-making process of firms considering whether to seek additional support from NPL. The underlying assumption is that firms will pursue further engagement if the marginal benefit they expect to receive exceeds a fixed marginal cost. The model is twofold:

The first component focuses on identifying the probability distribution that best represents the marginal value of an instance – or year – of support provided to firms. Through the application of the principle of maximum entropy, the analysis treats the population of firms as an isolated physical system, where the total "energy" or investment is fixed, but there are many possible ways to distribute this investment among individual firms. This analogy allows the model to use the exponential distribution to describe the marginal value of support, based on the idea that cumulative investment is a stock rather than a flow, and that the value of support can be redistributed among firms over time. By leveraging historical data, the report establishes a direct link between this probability distribution and the number of firms accessing support, which in turn enables the forecasting of engagement levels under different funding scenarios.

The second component of the model estimates NPL's stock of knowledge, which is a critical asset that depreciates over time unless actively maintained or enhanced through continuous R&D investment. This knowledge stock serves as a complementary input to the support NPL provides, directly influencing the expected value that recipient firms derive from their engagement. The model incorporates the dynamic relationships between R&D expenditure, the maintenance or growth of the knowledge stock, and the resulting utility of NPL's offerings to firms. Therefore, the framework provides a nuanced understanding of how changes in funding (both immediate and long-term) effect the ecosystem of firms that rely on NPL's expertise.

To illustrate the practical implications of this framework, the report presents three hypothetical scenarios: a 30% increase in annual R&D spending, a 30% decrease, and a one-year omission in R&D investment. Each scenario is analysed to show how variations in funding levels impact the knowledge stock and, consequently, the number of firms that benefit from NPL's support. The findings are clear: increased R&D investment leads to a larger and more robust knowledge stock, which in turn attracts greater engagement from firms. Conversely, reductions in funding result in a decline in both the knowledge stock and the number of supported firms. Even a temporary gap in funding produces measurable losses, though the analysis suggests that recovery is possible if investment resumes in a timely manner.

1 Introduction

1.1 Preface

The framework detailed in this report was developed to address a central question: *How does the extent of NPL's impact respond to changes in the inflow of public investment?* NPL plays a foundational role in enabling innovation, productivity, and competitiveness across sectors in the UK. However, the benefits of its work, particularly those stemming from sustained R&D, are often indirect, long-term, and difficult to measure using conventional evaluation methods.

Over the years, NPL has maintained a relatively stable level of R&D investment. As a result, its impact on firm engagement has reached a cyclical steady-state, where similar levels of investment tend to yield similar levels of engagement with user firms. This stability presents a unique opportunity – by analysing historical data, we have attempted to quantify the relationship between investment and impact, specifically the number of firms that engage with NPL's services. Using this framework, we can forecast future impact based on expected R&D expenditure.

The framework is designed to be practical and forward-looking. It is particularly useful during planning/development stages for making projections pertaining to future phases of existing programmes, e.g. the National Measurement System (NMS), or novel programmes altogether. By combining the results from this model with other economic models used by NPL (such as the NMS Business Case Model[1]), we can predict the Net Present Value and Value for Money for proposed R&D undertakings. The framework provides a basis for setting realistic benchmarks for monitoring and evaluation. Ultimately, this model is about turning abstract concepts like knowledge spillovers and long-term impact into tangible metrics that can guide strategy, investment, and evaluation.

1.2 Background

The National Physical Laboratory (NPL) is the United Kingdom's national metrology institute, responsible for developing and maintaining the country's measurement standards. As a public research institution, NPL supports innovation and productivity across the UK economy by providing scientific expertise, measurement services, and collaborative research opportunities to a wide range of firms and sectors. The impact of NPL's activities is not always immediately visible, as the benefits of research and development (R&D) investment often unfold over extended periods via complex interactions with industry.

The work of NPL is situated within the broader framework of the UK's National Measurement System (NMS), a coordinated network of laboratories and institutions that deliver measurement science to support innovation, trade, and regulation. Funded by the Department for Science, Innovation and Technology (DSIT), the NMS ensures that UK businesses have access to reliable and internationally recognised measurement standards. The NMS underpins critical sectors by reducing technical barriers to innovation and enabling firms to demonstrate compliance with global standards[2]. Through its collaborative model,

[1] King, M; Olakojo, S (2023) *NMS business case model: an explanatory note*. NPL Report. IEA 15

[2] Katanguru, D; Qureshi, Z; King, M (2023) *A survey of UK-based businesses using laboratories funded through the National Measurement System*. NPL Publications.

the NMS not only supports scientific excellence but also drives economic impact by enhancing productivity, facilitating market access, and fostering trust in emerging technologies. The strategic alignment of NPL's services with NMS objectives ensures that measurement science remains a foundational enabler of the UK's industrial and innovation strategies.

NPL has adopted a structured six-year model[3] to evaluate its impact. This model provides a systematic framework for assessing how NPL's R&D expenditure translates into tangible benefits for firms over rolling six-year periods. Central to this framework is the classification of firms based on the frequency and continuity of their interactions with NPL. Firms are grouped into three categories:

- "Regularly supported" or "Treated" (5 or more engagements over six years)
- "Close-to-Treated" (3 to 4 engagements over six years)
- "Pathway-to-Treated" (1 or 2 engagements over six years)

The six-year timeframe was chosen to reflect the typical lag between initial investment in scientific research and the realization of measurable outcomes in the wider economy. This period is long enough to capture the cumulative effects of knowledge creation, technology transfer, and firm engagement, while still allowing for regular review and adjustment of strategy.

Here, the definition of "engagement" is multifaceted. Firms may utilise NPL's expertise through direct collaborations on join projects, use of measurement services, or participation in knowledge transfer activities. Each of these interactions represents an "instance of support," and the model attempts to quantify both the number of firms benefiting from such support and the value they derive from it. Importantly, the model sits well with the understanding that the decision for a firm to seek support from NPL is influenced by a range of factors, including the perceived benefits, the costs involved, and the evolving stock of knowledge maintained by NPL.

2 Value of an Excludable Public Good

An excludable public good[4], also known as a club good, is a good or service that is non-rivalrous in consumption but excludable, meaning that it is possible to prevent non-paying individuals from using it, even if their use would not reduce the availability for others. NPL's utility as a national metrology institute stems from its stock of knowledge and capabilities, accumulated via years of public funding. As such, the nature of support offered by it to user firms in the form of collaboration, service, or knowledge transfer, is rendered non-rivalrous. The barrier that makes it excludable is the cost – associated with seeking said forms of support – borne by its users. We treat NPL's userbase of firms like an isolated physical system (more on this below) with a constant number of potential user firms, $\mathbb{N}$ indexed by $i \in \{1, 2, \dots, \mathbb{N}\}$.

Over a six-year period, different companies in this userbase access support a varying number of times. Some might use it once, others multiple times, and some not at all. We

[3] Dias, C; King, M (2023) *Growth and Survival of Supported Firms*. NPL Publications.

[4] Samuelson, Paul (1954). *The Pure Theory of Public Expenditure*. The Review of Economics and Statistics. 36 (4): 387–389. doi:10.2307/1925895. JSTOR 1925895

group them based on how many times they've accessed support. For example, one group includes companies that used support once, another group includes those that used it twice, and so on, up to six times. We then count how many companies fall into each group. This helps NPL track usage patterns across its entire user base. Interestingly, the number of companies in each group has remained stable since 2013 (which is as far back as our data on firm interaction goes). This means that the way companies use NPL's support hasn't changed much over time. Let q_i be number of instances (years) of support used by firm $i \in \{1, 2, \dots, \mathbb{N}\}$ during a 6-year period and, therefore, $q_i \in \{0, 1, 2, \dots, 6\}$. If n_q denotes the number of firms with q instances of support, we get $N = \sum_{q=0}^{6} n_q$ (not to be confused with $\mathbb{N}$, which is the fixed total number of firms in the userbase). We are assuming that the values of n_q have reached a steady state. As depicted by the chart below, the average of n_q has remained stable since 2013 for all values of q.

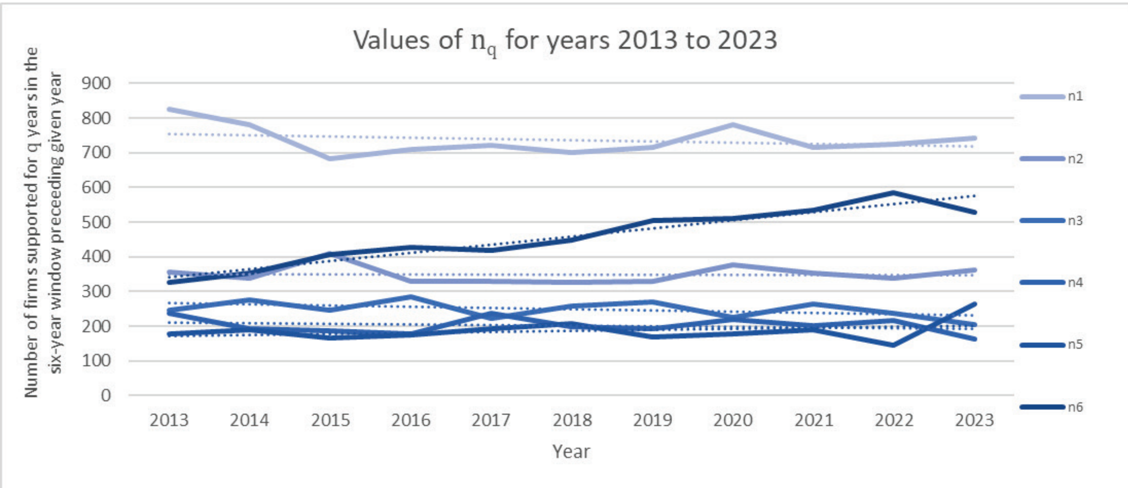

Figure 1 Values of n_q for each year between 2013 to 2023. This graph excludes n_0 for ease of observation. Please refer to Annex B for the data table.

Now, not every company gains the same value from using NPL's support. Some benefit a lot, while others benefit less. We call the total value a company gets from using support its benefit. If we add up the benefits for all companies, we get the total value of support delivered by NPL. Let $v_i(q)$ be the benefit enjoyed by firm i from q instances of support. The aggregate value of the support delivered is $V(\boldsymbol{q}) = \sum_{i=1}^{\mathbb{N}} v_i(q)$. Let $v_i'(q) = \frac{dv_i}{dq}$ denote the marginal benefit from the latest instance of support.

The cost borne by a firm to access an instance of support includes fees paid to NPL (if any) and the money spent by a firm internally on allocation of internal resources, time, strategic focus, etc. We assume that on average this cost is the same for all firms. We denote it by c and estimate the value to be £167,906 (details on this estimation have been provided in Annex C Table 6). When deciding whether to use NPL's support again, companies weigh the extra benefit they expect to receive against the cost of accessing that support. If the expected benefit from one more instance of support is greater than the cost, they'll choose to use it again. That is, a firm will want to access the q^{th} instance of support if $'(q) > c$.

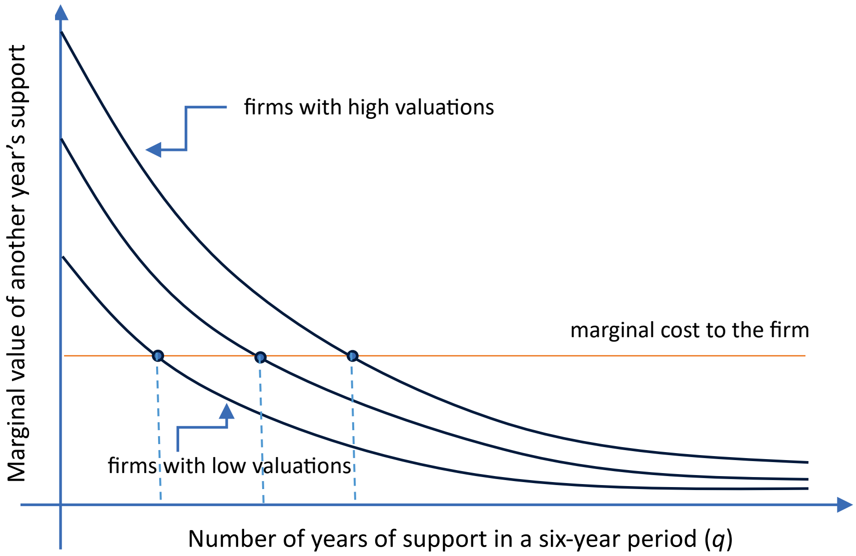

Figure 2 Marginal value vs number of instances of support for different types of firms

3 Principle of Maximum Entropy

This section explains how we may use the "principle of maximum entropy" (Botev and Kroese, 2008 & 2011) to ascertain the probability distribution which best represents the marginal benefit of an extra instance of support, $v'(q)$. We do this in order to estimate the probability that this marginal benefit exceeds the marginal cost, $\mathrm{P}(v'(q) \geq c)$.

3.1 Value of NPL's knowledge

Knowledge as a good differs from typical capital due to its non-rivalrous nature. Multiple firms can benefit from the same stock of knowledge[5] simultaneously. Correspondingly, the value of publicly available knowledge is the sum of willingness-to-pay of all its users (firms)[6]. Here, this means the aggregate value ascribed by a group of firms to being able to access NPL's knowledge through the support it provides to them.

Another insight is that the value of NPL's support depends on the stock of knowledge that it has accumulated by continuously investing in R&D[7]. This knowledge stock is complementary to the investments in tangible and intangible assets made by the firms in its userbase. Hence, NPL's support creates benefits for these firms by augmenting the value of their own prior investments in equipment, skills, and capabilities. In particular, the complementary assets owned by the firms could be instrumentation and measurement capabilities.

The situation can be formalised as follows: Suppose that a set of $\mathcal{N}$ firms has benefited from NPL's support for the last $q-1$ years and now has the option of a further year of support. That is, these $\mathcal{N}$ firms have the option of a q^{th} year of support. Let K_{NPL} denote the stock of

[5] Knowledge capital refers to the intangible assets of an organization encompassing the collective knowledge, skills, experience, and information held by individuals and embedded within organizational processes, systems, and relationships.

[6] Stiglitz, J. E. (1999). *Knowledge as a Global Public Good.*

[7] Akcigit, U., Celik, M. A., & Greenwood, J. (2013). *Buy, Keep or Sell: Economic Growth and the Market for Ideas* (NBER Working Paper No. 19763). National Bureau of Economic Research

NPL's knowledge (see Section 4 below) and k_i denote the value of complementary assets owned by the i^{th} firm. Lastly, the value of the assets is the present value of the flow of profits they generate for the firms.

It is reasonable to suppose that a firm's assets appreciate over time due to two main reasons: firstly, human capital (skills) and knowledge assets both appreciate with use. Secondly, some of the firm's profits are reinvested each year in new and improved capabilities. Consequently, the value of the assets under the firm's control will tend to grow each year. Moreover, access to the knowledge that's embodied in NPL's services, enables the firms to accelerate the rate at which their own private assets appreciate.

Let time t be measured in years and use Newton's dot notation for derivatives with respect to time. Thus, the yearly increase in the value of their assets is $\dot{k} := dk/dt$, and their corresponding growth rate is $g := \dot{k}/k$.

If the i^{th} firm accesses another year of support from NPL, it contributes to the growth in the value their assets. The increase in value of k_i that occurs in unit time (a year) is given by $\dot{k}_i = \alpha k_i + \theta k_i K_{NPL}$, where α and θ are to be regarded as constant parameters. In terms, of the growth rate this can be expressed as $g_i = \alpha + \theta K_{NPL}$[8], which implies that α is the baseline growth rate. In other words, the first term of this equation is intended to represent the growth that isn't attributable to NPL. Hence, if all N firms access another year of NPL's support, the aggregate increase in wealth attributable to NPL's knowledge becomes:

$$\sum_{i=1}^{N} \dot{k}_i - \alpha\left(\sum_{i=1}^{N} k_i\right) = \theta\left(\sum_{i=1}^{N} k_i\right) K_{NPL} \qquad 3.1$$

3.2 Probability distribution of the marginal benefit

Since NPL's knowledge is non-rivalrous but excludable, firms wishing to benefit from it must pay an "access fee". This section adapts a classic derivation of the Maxwell-Boltzmann distribution (1860), for the energies of particles in a gas, to motivate an exponential demand function for firms using NPL's support. The following analysis shows that the exponential distribution satisfies the maximum entropy principle in a situation where the number of firms who could benefit from support is fixed, and the total value ascribed by these firms to the support is also fixed. This approach is based on the concept of entropy as used in information theory and statistical mechanics. The two foundational references for this kind of approach are:

- Jaynes, E.T. (1957). "Information theory and statistical mechanics" (PDF). Physical Review. 106 (4): 620–630. Bibcode:1957PhRv..106..620J. doi:10.1103/PhysRev.106.620. S2CID 17870175.
- Jaynes, E.T. (1957). "Information theory and statistical mechanics II" (PDF). Physical Review. 108 (2): 171–190. Bibcode:1957PhRv..108..171J. doi:10.1103/PhysRev.108.171.

Specifically, the Principle of Maximum Entropy says that the preferred distribution is the one that maximises the Shannon information entropy (a measure of the average uncertainty or randomness), subject to whatever constraints imposed by macroscopic information about the state of the system. In simpler terms, the probability distribution that best represents the

[8] $g_i = \dot{k}_i/k = \alpha k_i/k + \theta(k_i/k)K_{NPL}$

current state of knowledge is the one with the most uncertainty, given the available information. This line of reasoning has its origins in the groundbreaking work by Boltzmann, Maxwell, and Gibbs in the field of Thermodynamics. However, the maximum entropy approach is a general technique for statistical inference with applications far beyond its origins:

Similar to an isolated physical system with a fixed and conserved aggregate energy, the population of firms who could benefit from NPL's support can be seen as an isolated system – the conserved quantity being the cumulative investment in assets, both tangible and intangible. In our context, firms are analogous to particles in system and the valuation each firm assigns to NPL support is analogous to the energy of a particle. The principal simplifying assumption is that as these assets constitute a stock (as opposed to a flow), this quantity has a set value at any point in time. This mirrors the conservation of energy in physics – a key requirement for applying Boltzmann entropy. These assets could be redistributed (exchanged) in various ways amongst the set of firms in the population, but their aggregate wealth is fixed by the amount of investment that took place in the past.

Next, it's possible for any system composed of lots of underlying constituents (atoms with energies or firms with assets) to be in wide range of different states depending on how these constituents arrange themselves. Hence, it's helpful to begin by introducing the idea of a statistical ensemble, which is the probability distribution over all the possible states that the system could be in. The distribution of valuations across firms is the macrostate, whereas the number of ways firms can be arranged into valuation bands is the number of microstates. This approach is mathematically rigorous and aligns with previously published economic models[9].

The microscopic states of the system need to be compatible with the macroscopic constraints. In the absence of other information, the microscopic states compatible with these constraints should be assigned equal probabilities. This is in line with the 'equal a priori probability postulate' which states that if an isolated system has a fixed energy and other conserved quantities, it is equally likely to be found in any of its accessible microstates – the probability of any one microscopic state being the reciprocal of the total number of compatible microstates.

Let W denote the number of ways that the microscopic constituents of a system can be arranged to yield a certain macroscopic state. Boltzmann's principle states that the entropy, S, of a macroscopic state is proportional to the logarithm of number of ways that it can come about through different arrangements of the microscopic constituents: $S = k_B \ln(W)$, where k_B is a constant of proportionality[10]. In the current context, these constituents are the firms and the values they ascribe to the support.

Suppose that $\mathcal{N}$ firms have already had $q - 1$ years of support from NPL. Each of these firms ascribe some non-negative value to one further year of support. Each such firm ascribes their own value to the q^{th} year of support. The range for these valuations extends

[9] Yakovenko, 2012. *Applications of statistical mechanics to economics: Entropic origin of the probability distributions of money, income, and energy consumption*. Papers 1204.6483, arXiv.org.

Rosser JB Jr. *Econophysics and the Entropic Foundations of Economics.* Entropy (Basel). 2021 Sep 30;23(10):1286. doi: 10.3390/e23101286.

[10] Boltzmann constant.

from the minimum to the maximum, which can then be divided into M intervals of equal length. Let n_j the number of firms in the j^{th} band, where $\sum_{j=1}^{M} n_j = \mathcal{N}$.

Let us refer to the arrangement of number of firms in each band as the macroscopic state and recognise that different microscopic states can yield the same macroscopic state. Next, consider the number of ways that $\mathcal{N}$ firms can be put into M bands to give the same macroscopic state, without worrying about the arrangement of the firms within the bands which are indistinguishable in terms of the macroscopic state of the system. With $\mathcal{N}$ balls there are $\mathcal{N}!$ possible arrangements but with M boxes there are only the following possible number of distinguishable arrangements[11] -

$$W = \frac{\mathcal{N}!}{n_1!\, n_2! \cdots n_M!}$$

The probability of any one of the arrangements becomes $P = 1/W$. Since, the entropy of a given macroscopic state is proportional to the log of the number of compatible permutations:

$$\ln(W) = \ln\left(\frac{\mathcal{N}!}{n_1!\, n_2! \cdots n_M!}\right) = \ln(\mathcal{N}!) - \sum_{j=1}^{M} \ln(n_j!)$$

Stirling's approximation for a factorial[12] is $ln(n!) \approx n \times ln(n) - n$, from which we get the following expression:

$$\mathbf{\ln(W) = \mathcal{N}.\ln(\mathcal{N}) - \mathcal{N} - \sum_{j=1}^{M} n_j.\ln(n_j)} \qquad 3.2$$

Since $\mathcal{N}$ is a fixed parameter, maximising $\ln(W)$ is equivalent to minimising $\sum_{j=1}^{M} n_j \ln(n_j)$.

Consider a firm's willingness-to-pay for access to NPL's knowledge. This corresponds to the aforementioned attributable growth in the value of a firm's assets, $\dot{k}_i$. The next step is to turn this continuous variable into a discrete variable. This is clearly an approximation, and we are free to choose the intervals in a way that aids the analysis.

Let y_j denote the midpoint of the j^{th} band. Our analysis will be simplified by assuming that all firms in the j^{th} band take y_j as their valuation. To be even more specific suppose that:

$$\mathbf{y_1 = 0;\ y_2 = 1;\ y_3 = 2;\ \ldots\ ;\ y_M = M - 1.} \qquad 3.3$$

Moreover, suppose that the average, $\bar{y} > 1$.

Thus, for the $\mathcal{N}$ firms, the average value of the valuations is given by $\bar{y} = \frac{1}{\mathcal{N}} \sum_{j=1}^{M} y_j n_j$. This implies that the value delivered if all $\mathcal{N}$ firms received the q^{th} year of support would be $Y = \mathcal{N}\bar{y}$, and so this corresponds to the total value amongst the population of firms. It is also

[11] When arranging $\mathcal{N}$ unique balls into M boxes, some arrangements look the same if balls within a box are swapped. To count only distinct groupings, we divide the total arrangements ($\mathcal{N}!$) by all possible internal arrangements within each box ($n_1!, n_2!, \ldots, n_m!$).

[12] Stirling, *Methodus Differentialis.* 1730.

assumed that the total value is fixed by the size of the knowledge stock as represented by the cumulative (depreciated) sum of annual R&D spending.

Now, suppose that the following elements have already been determined by the cumulative investments in R&D, technical skills, and capital equipment:

- $\mathcal{N}$ is the number of firms who have already had q instance of support and could benefit from further support (population).
- $Y = \mathcal{N}\bar{y}$ is the total value of delivering the q^{th} year of support to each firm in the population.

The idea is that Y is fixed by the tangible and intangible assets that have been built up through a mix of public and private investment. If $\mathcal{N}$ and Y are fixed, we need to determine the arrangement that maximises the entropy subject to the constraints. The problem can be set up as follows:

- The objective function is $f(n) = \sum_{j=1}^{M} n_j \ln(n_j)$.
- Since $\mathcal{N}$ is fixed, the first constraint is $g(n) = \mathcal{N} - \sum_{j=1}^{M} n_j$.
- Since Y is fixed, the second constraint is $h(n) = Y - \sum_{j=1}^{M} n_j y_j$, where y_j are parameters.

$$\min_{n_j} f(n) \text{ such that } g(n) = 0 \text{ and } h(n) = 0$$

This problem can be solved by introducing Lagrange multipliers[13]:

$$\frac{\partial f}{\partial n_i} + \lambda \frac{\partial g}{\partial n_i} + \mu \frac{\partial h}{\partial n_i} = 0,$$

where λ is the multiplier for the 1st constraint, and μ is the multiplier for the 2nd constraint. Evaluating these partial derivatives gives $1 + \ln(n_j) - \lambda - \mu y_j = 0$, which then implies:

$$n_j = a.\exp(\mu y_j),$$

where a is a constant. The next step is to find a way of determining a and μ using the two constraints.

$$\sum_{j=1}^{M} a.\exp(\mu y_j) = \mathcal{N}$$

$$\sum_{j=1}^{M} a y_j.\exp(\mu y_j) = Y$$

Denoting $R = \exp(\mu)$ and using Equation 3.3 these constraints can be rewritten as follows:

[13] Lagrange multipliers are used for finding the local maxima and minima of a function when equality constraints are involved. The method introduces new variables called Lagrange multipliers, one for each constraint, and creates a new function called the Lagrangian – a single expression which combines the objective function and the constraints.

$$1 + R + R^2 + \cdots + R^{M-1} = \mathcal{N}/a$$

$$R + 2R^2 + 3R^3 + \cdots + (M-1)R^{M-1} = Y/a$$

Notice the lefthand side of the 1st constraint is a geometric series. Furthermore, the lefthand side of the 2nd constraint equals the first derivative of the first series multiplied by R. If M is suitably large and R is suitably small ($0 < R < 1$), then the sum of the M terms in the series can be approximated using the formula for an infinite geometric series[14]:

$$1 + R + R^2 + \cdots + R^{M-1} \approx \frac{1}{1-R}$$

Furthermore, by differentiating and multiplying through by R, this then implies that:

$$R + 2R^2 + 3R^3 + \cdots + (M-1)R^{M-1} \approx \frac{R}{(1-R)^2}$$

Using these approximations the two constraints can be rewritten as follows:

$$\frac{1}{1-R} = \frac{\mathcal{N}}{a}$$

$$\frac{R}{(1-R)^2} = \frac{Y}{a}$$

Dividing the 2nd constraint by the 1st constraint eliminates a:

$$\frac{R}{1-R} = \frac{Y}{\mathcal{N}} = \bar{y}$$

Solving for R gives:

$$R = \frac{\bar{y}}{1+\bar{y}}$$

Notice that $0 < R < 1$ providing that $\bar{y} > 0$. Since $R = \exp(\mu)$, this can be rewritten as:

$$\exp(\mu) = \frac{\bar{y}}{1+\bar{y}}$$

Taking the reciprocal of both sides, this becomes:

$$\exp(-\mu) = \frac{1+\bar{y}}{\bar{y}}$$

Taking the natural log of both sides gives:

[14] If $|r| < 1$, the terms of the series approach zero (becoming smaller and smaller in magnitude) and the sequence of partial sums S_n converge to a limit value of $\frac{a}{1-r}$.

$$-\mu = \ln(1+\bar{y}) - \ln(\bar{y}) \approx \frac{1}{\bar{y}}$$

The approximation here is based on the calculus properties of the natural log function[15]. Hence, the key result for μ is as follows:

$$\mu = -\frac{1}{\bar{y}}$$

This implies that $R = \exp(-1/\bar{y})$; and from the 1st constraint, this implies that:

$$\frac{1}{1-\exp(-1/\bar{y})} = \frac{\mathcal{N}}{a}$$

Rearrangement then gives:

$$a = \mathcal{N}[1-\exp(-1/\bar{y})]$$

Furthermore, if $\bar{y} > 1$, then $1-\exp(-1/\bar{y}) \approx 1-(1-1/\bar{y}+\cdots) = 1/\bar{y}$, which implies the following result for a:

$$a = \mathcal{N}/\bar{y}$$

Substituting these results for μ and a back into the original constraint equations gives us:

$$\sum_{j=1}^{M} (\mathcal{N}/\bar{y}).\exp\left(-y_j/\bar{y}\right) = \mathcal{N}$$

$$\sum_{j=1}^{M} \left(\mathcal{N}y_j/\bar{y}\right).\exp\left(-y_j/\bar{y}\right) = Y$$

A little simplification then yields:

$$\sum_{j=1}^{M} (1/\bar{y}).\exp\left(-y_j/\bar{y}\right) = 1$$

$$\sum_{j=1}^{M} \left(y_j/\bar{y}\right).\exp\left(-y_j/\bar{y}\right) = \bar{y}$$

These equations show that $(1/\bar{y}).\exp\left(-y_j/\bar{y}\right)$ can be interpreted as the probability density function for y_j.

Shannon entropy is defined for discrete random variables, while the differential entropy (also called continuous entropy) is its counterpart for continuous random variables. A continuous approach for this derivation can be found in Annex D.

[15] $\Delta\ln(x)/\Delta x \approx d\ln(x)/dx = 1/x$

3.3 Expected value of marginal benefit

Upon, combining the above result with terminology used in Section 2.2. The probability of the marginal benefit exceeding the marginal cost, $\mathrm{P}(v'(q) \geq c)$ can be depicted by the following graph (shaded region):

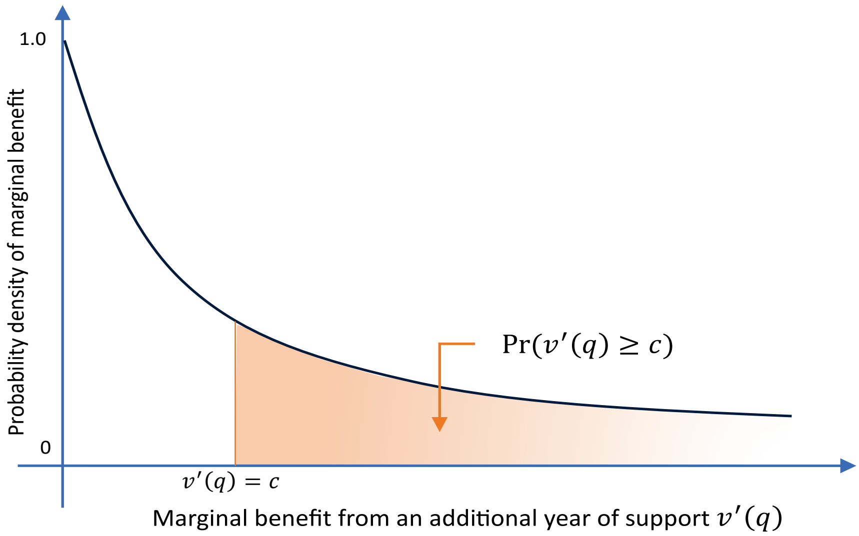

Figure 3 Probability of marginal benefit from an instance of support exceeding marginal cost of accessing the support

We look at how many firms have received at least q instances of support compared to those that have received at least $q-1$ instances. The ratio of these two quantities reflects the proportion of firms that, having already engaged $q-1$ times, chose to engage once more – implying that their expected marginal benefit exceeded the cost. This ratio serves as a proxy for the probability that a firm continues its engagement with NPL, and it allows the model to estimate the expected marginal benefit $\mathbb{E}[v'(q)]$ using the properties of the exponential distribution. It links observable firm behaviour directly to the underlying economic decision-making process, without requiring detailed data on individual firm valuations or costs.

It can be denoted as:

$$\mathbf{P}(\boldsymbol{v}'(\boldsymbol{q}) \geq \boldsymbol{c} \mid \boldsymbol{N}(\boldsymbol{q}-\mathbf{1})) = \frac{\boldsymbol{N}(\boldsymbol{q})}{\boldsymbol{N}(\boldsymbol{q}-\mathbf{1})} \quad 3.4$$

Where,

- $N(q)$ is the number of firms with q or more instances of support.
- $N(q-1)$ is the number of firms with $q-1$ or more instances of support

Here, $N(q) = \sum_{k=q}^{6} n_k$, not to be confused with $\mathbb{N}$, which refers to the total number of firms in the user base.

Since we know that $v'(q)$ follows the exponential distribution with parameter $\mathbb{E}[v'(q)]$, we get the following complimentary cumulative distribution:

$$\mathbf{Pr}(\boldsymbol{v}'(\boldsymbol{q}) \geq \boldsymbol{c}) = \mathbf{exp}(-\boldsymbol{c}/\mathbb{E}[\boldsymbol{v}'(\boldsymbol{q})]) \quad 3.5$$

Upon combining equations 3.4 and 3.5 above, we get the following:

$$\mathbb{E}[\boldsymbol{v}'(\boldsymbol{q})] = -\boldsymbol{c}/\mathbf{ln}[\boldsymbol{N}(\boldsymbol{q})/\boldsymbol{N}(\boldsymbol{q}-\mathbf{1})] \quad 3.6$$

This key equation gives us a way to connect the marginal benefit from q^{th} instance of support and engagement data with supported firms. We can use one side of the equation to estimate the other, based on the type of data available.

4 The Knowledge Stock

We know that there is direct relationship between the stock of knowledge capital associated with a public good and the value of benefit it provides to its users, as detailed above in Section 3.1. This section attempts to ascertain the value from NPL's support using its estimated stock of knowledge.

4.1 Knowledge stock versus R&D spend

First, we utilise a knowledge accumulation model[16] to establish an equation between R&D expenditure and stock of knowledge in the context of NPL. NPL's knowledge is like a reservoir that fills up with each year of investment but also slowly leaks due to depreciation. If NPL keeps investing consistently, the reservoir stays full and stable. But if investment increases, the reservoir fills faster and reaches a higher level; if investment drops or stops, the reservoir shrinks.

If t denotes time in years,

- Let $R(t)$ denote the flow of R&D spending in year t.
- Let $K(t)$ denote the stock of knowledge in year t. This is the cumulative R&D spending minus losses from depreciation.

Suppose δ is the rate at which knowledge assets depreciate. According to ONS, the depreciation rate is 15% for R&D investments. A portion of previous year's knowledge is retained after accounting for depreciation, and new R&D investment adds to it. The evolution of the knowledge stock is given by the following equation:

$$\frac{\boldsymbol{dK}(\boldsymbol{t})}{\boldsymbol{dt}} = \boldsymbol{R}(\boldsymbol{t}) - \boldsymbol{\delta K}(\boldsymbol{t}-\mathbf{1}) \quad 4.1$$

The following equation models how NPL's knowledge stock grows over time when R&D spending is constant, $\bar{R}$. The first term shows how the initial knowledge stock, $K(0)$, decays due to depreciation. The second term shows how new investment gradually builds up the

[16] Prendergast, R. (2010). *Accumulation of knowledge and accumulation of capital in early 'theories' of growth and development*. Cambridge Journal of Economics, 34(3), 413–431.

stock. Over time, the effect of the initial stock fades, and the system stabilises at a new level determined by the rate of investment and depreciation.

$$\boldsymbol{K(t) = K(0)\exp(-\delta t) + \frac{\bar{R}}{\delta}[1 - \exp(-\delta t)]} \qquad 4.2$$

As time goes on, i.e., $t \to \infty$, the result asymptotes towards a steady state. If NPL keeps investing the same amount every year, the knowledge stock eventually settles at a fixed level – the point where inflow and outflow balance out.

$$\boldsymbol{K(t) \to K^*}, \text{ where } \boldsymbol{K^* = (\bar{R}/\delta)} \qquad 4.3$$

Let us now consider the effect of a change in the annual flow of R&D spending. Let $t = 0$ denote the present. Suppose that for $t < 0$ (in the past), the annual flow of R&D was $\bar{R}$, whereas for $t \geq 0$, the annual flow of R&D becomes $\hat{R}$. In this situation, the knowledge stock becomes:

$$\boldsymbol{K(t) = \frac{\bar{R}}{\delta}\exp(-\delta t) + \frac{\hat{R}}{\delta}[1 - \exp(-\delta t)]} \qquad 4.4$$

The first term represents the leftover knowledge from the previous investment rate $\bar{R}$, which gradually fades due to depreciation. The second term shows the build-up of new knowledge from the new investment rate $\hat{R}$, which grows over time and eventually dominates. Notice that if $\hat{R} > \bar{R}$, then the stock increases. As time progresses, the influence of the old investment diminishes, and the knowledge stock converges to a new steady state determined by $(\hat{R}/\delta)$. This model helps us understand how quickly NPL can recover or grow its knowledge base after a funding change, and how that affects its ability to support firms.

Supposing that NPL started from scratch a long time ago and then consistently* spent £48.3 million per year on R&D[17]. This implies that $K(0) = 0$, $\bar{R} = £48.3m$, and according to ONS the depreciation rate for R&D investments is 15% per year[18]. Plugging these numbers into Equation 4.3 leads to the following value for the equilibrium knowledge stock: $K^* = £48.3m/15\% = £322m$.

4.2 Knowledge stock versus expected marginal benefit

NPL's knowledge stock directly affects the value of benefit gained by a firm when receiving support from NPL. Similar to how improved public amenities lead to higher property value, information, skills, expertise, and intellectual resources accumulated by NPL enable and

[17] This figure is the average of the latest available data (2017-2023) as stated in the Annual Accounts (AA) reports submitted to FAME-Orbis, and deflated according to *GDP deflators at market prices, and money GDP June 2025* (Quarterly National Accounts).

* This is assumption is difficult to justify in a robust manner due to limited availability of historical data. Available figures fall in the range of approximately ± £5 million around the average (Standard Deviation = £5.26*m*). The dataset is too small for us to ascribe a trend and, therefore, we have proceeded with the assumption of a constant value R&D expenditure across years.

[18] Office for National Statistics. *Experimental estimates of investment in intangible assets in the UK*. 2015

enhance private industry assets by serving as a foundation for innovation, signalling credibility, and reducing information asymmetry[19].

Let $\phi(q)$ denote the factor quantifying the relationship between NPL's knowledge stock, K, and the expected value from an instance of support, $\mathbb{E}[v'(q)]$. It signifies the increase in expected value of the q^{th} instance of support that comes from increasing K by £1m, where $\phi(q)$ is a factor indexed by q. This implies:

$$\mathbb{E}[\boldsymbol{v}'(\boldsymbol{q})] = \boldsymbol{\phi}(\boldsymbol{q})\boldsymbol{K} \tag{4.5}$$

We can determine the values of $\phi(q)$, $q \in \{0,1,2,\dots,6\}$ by merging equations 3.6 and 4.5.

$$\boldsymbol{\phi}(\boldsymbol{q}) = \frac{-\boldsymbol{c}}{\boldsymbol{K}.\ln[\boldsymbol{N}(\boldsymbol{q})/\boldsymbol{N}(\boldsymbol{q}-\mathbf{1})]} \tag{4.6}$$

We input the average values of $N(q) = \sum_{k=q}^{6} n_k$ calculated using historical data between years 2013-2023 (tabulated in Table 8 in the Annex) and our estimated value of c into this equation above. This gives us the following values of $\phi(q)$:

q	Average n_q	$P(v'(q) \geq c) = \frac{\text{Average } N(q)}{\text{Average } N(q-1)}$	$\mathbb{E}[v'(q)]$	$\phi(q)$
0	3433	-	-	-
1	737	0.3886	£177,641	5.52 x10^{-4}
2	350	0.6622	£407,408	1.27 x10^{-3}
3	248	0.7578	£605,384	1.88 x10^{-3}
4	202	0.7735	£653,817	2.03 x10^{-3}
5	187	0.7615	£616,282	1.91 x10^{-3}
6	458	0.7101	£490,407	1.52 x10^{-3}

Table 1 Estimated values of $\phi(q)$ based on historical data

5 Forecasting Future Impact

Using the values of parameter $\phi(q)$ established above, we can forecast the number of firms with $q \in \{1,2,\dots,6\}$ years of support in the future based on changes in funding (R&D spend).

i. Any change in the yearly R&D expenditure will translate into a change in NPL's stock of knowledge. Using Equation 4.1, we get –

[19] Ashish Arora & Sharon Belenzon & Bernardo Dionisi, 2023. *First-mover advantage and the private value of public science*. Research Policy, vol 52(9)

Carnabuci, Gianluca & Operti, Elisa. (2014). *Public Knowledge, Private Gain: The Effect of Spillover Networks on Firms' Innovative Performance*. Journal of Management. 10.1177/0149206311422448.

$$\boldsymbol{K(t) = (1 - \delta) \times K(t-1) + R(t)} \quad 5.1$$

ii. This change in the knowledge stock affects the expected value that a firm will derive from an additional instance of NPL's support. $\mathbb{E}[v'(q)]$ can be forecasted using Equation 4.5 and values of parameter $\phi(q)$ obtained above.

iii. We know that $v'(q)$ follows the exponential distribution with $\mathbb{E}[v'(q)]$ being the scale parameter. The probability of a firm deciding to access another instance of support, i.e., probability of marginal benefit $v'(q)$ exceeding the marginal cost c, is the complimentary cumulative function of this distribution and can be estimated using Equation 3.4.

iv. Since these probabilities represent the ratio of number of firms with q or more instances of support to the number of firms with at least $q-1$ instances of support, Equation 3.5 helps us estimate values of $N(q) = N(q-1) \times \Pr(v'(q) \geq c)$; and $n_q = N(q-1) - N(q)$ for $q \in \{1, 2, \dots, 6\}$.

6 Illustrative Examples

6.0 No change in R&D expenditure

Before exploring the effects of changes in the annual R&D budget, we establish a baseline by assuming no change, i.e., continued investment of £48.3*m*. The knowledge stock would remain the same as the current value of $K(t) = K^* = £322m$. Therefore, figures for *Pathway-to-*, *Close-to-*, and *Treated* firms will follow the steady state pattern tabulated in Table 8 –

Year	Pathway-to-Treated	Close-to-Treated	Treated
t-1	1087	450	645
t	1087	450	645
t+1	1087	450	645
…	…	…	…
t→∞	**1087**	**450**	**645**

Table 2 Status of supported firms over the years for Scenario 6.0.

6.1 Increase in R&D expenditure

Consider a scenario where NPL sees a 30% increase in its yearly R&D expenditure, i.e., $R(t) = £62.78\,million$ being the new yearly spend. The overall knowledge stock would see a jump and a gradual increase, eventually plateauing at $K(t \to \infty) = \frac{£62.78m}{0.15} = £418.5\,million$ (using Eq 4.3).

Number of firms with q instances of support, n_q, will also follow suit over time after the increase in R&D spend. Yearly figures for *Pathway-to-*, *Close-to-*, and *Treated* firms are, therefore, estimated to become as follows –

Year	Pathway-to-Treated	Close-to-Treated	Treated
t-1	1087	450	645
t	1098	467	708
t+1	1104	480	762
t+2	1109	490	808
t+3	1113	498	846
t+4	1114	504	880
t+5	1116	510	906
t→∞	**1117**	**534**	**1063**

Table 3 Status of supported firms over the years for Scenario 6.1. See Annex F for background calculations.

6.2 Decrease in R&D expenditure

Conversely, if NPL experienced a 30% dip in its yearly R&D spend, i.e., $R(t) = £33.8\,million$, the overall knowledge stock would decline gradually, eventually reaching a steady state at $K(t \rightarrow \infty) = \frac{£33.8m}{0.15} = £225.3\,million$ (using Eq 4.3).

Number of firms with q instances of support, n_q, will also follow suit over time after the decrease in R&D spend. Yearly figures for *Pathway-to-*, *Close-to-*, and *Treated* firms are, therefore, estimated to become as follows –

Year	Pathway-to-Treated	Close-to-Treated	Treated
t-1	1087	450	645
t	1073	431	583
t+1	1059	414	530
t+2	1044	398	486
t+3	1030	384	449
t+4	1017	371	418
t+5	1004	360	392
t→∞	**911**	**289**	**255**

Table 4 Status of supported firms over the years for Scenario 6.2. See Annex F for background calculations.

6.3 One-off dip in R&D expenditure

We also consider a scenario wherein the annual R&D spend stays the same but skips a year (year t). This is to estimate the opportunity lost from a gap in R&D activities. A year of no funding would mean a drop in the knowledge stock to $K(t) = £273.6\, million$. With time, parallel to this knowledge stock, the values of n_q will rise and eventually converge with the steady state figures detailed in Table 1. However, the immediate future will look like as follows –

Year	Pathway-to-Treated	Close-to-Treated	Treated
t-1	1087	450	645
t	1027	380	440
t+1	1039	392	470
t+2	1047	402	496
t+3	1054	410	518
t+4	1061	417	536
t+5	1065	422	553
t→∞	**1087**	**450**	**645**

Table 5 Status of supported firms over the years for Scenario 6.3. See Annex F for background calculations.

In the absence of a year's funding, we immediately risk losing 50 *Pathway-to-Treated* firms, 70 *Close-to-Treated* firms, and 205 *Treated* firms. NPL will gradually gain them back, provided it returns to spending consistently on R&D.

6.4 Summary charts

The following charts depict how the changes in number of supported firms flow parallel to those in the stock of knowledge under each scenario.

Knowledge Stock

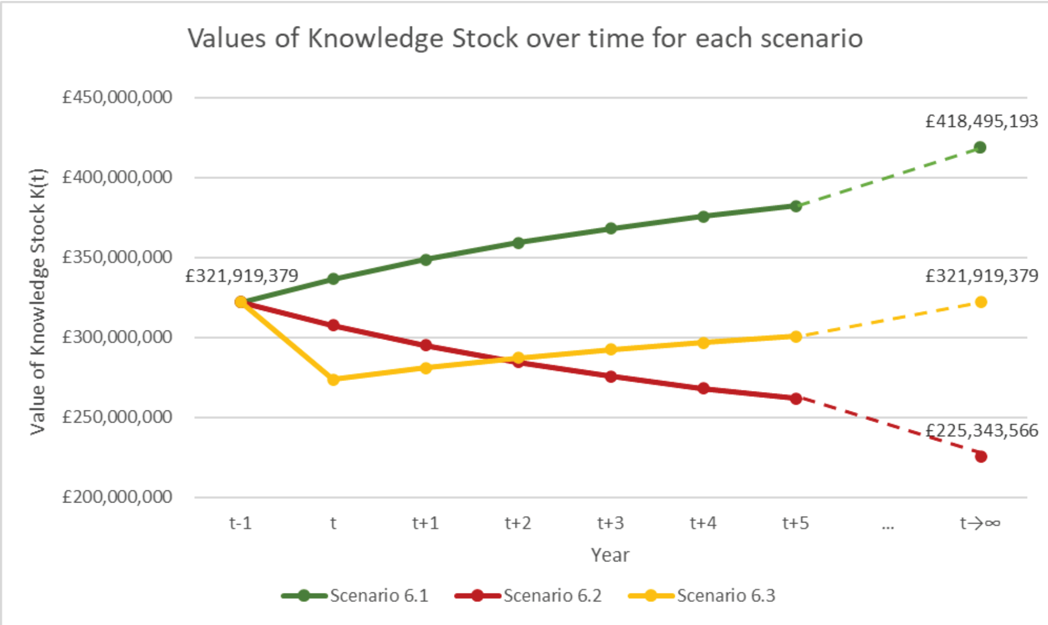

Figure 4 Evolution of knowledge stock $K(t)$ for each scenario

Supported Firms

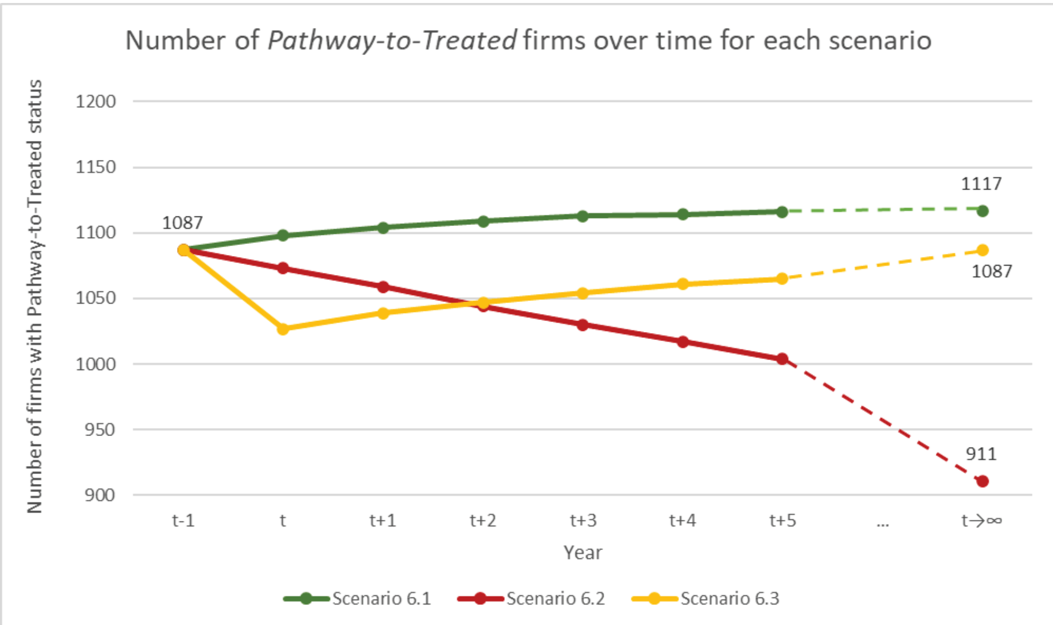

Figure 5 Change in number of *Pathway-to-Treated* firms $(n_1 + n_2)$ in each scenario

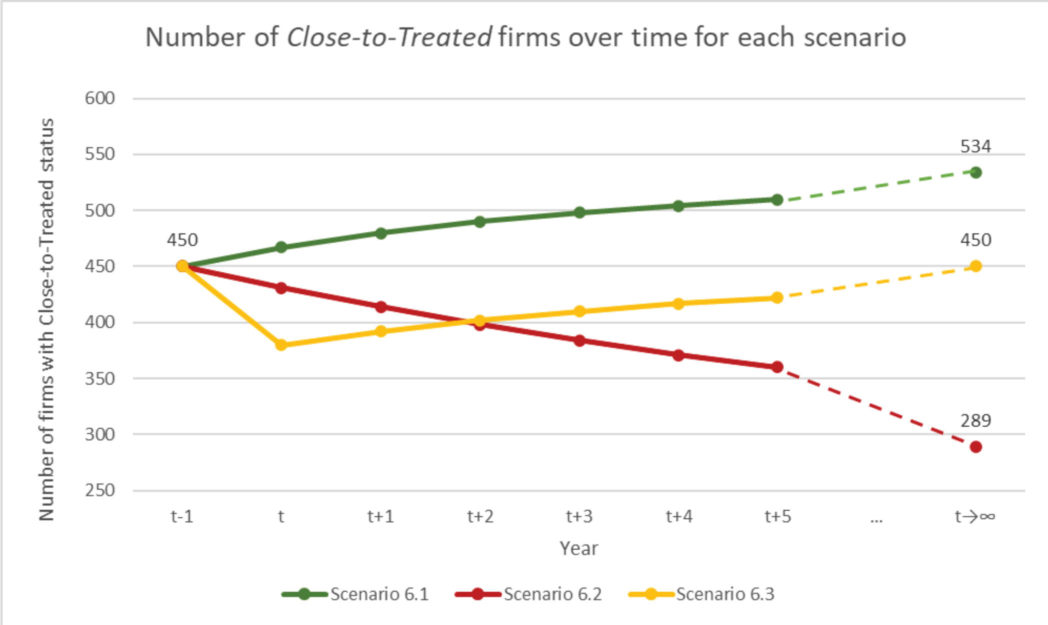

Figure 6 Change in number of *Close-to-Treated* firms ($n_3 + n_4$) in each scenario

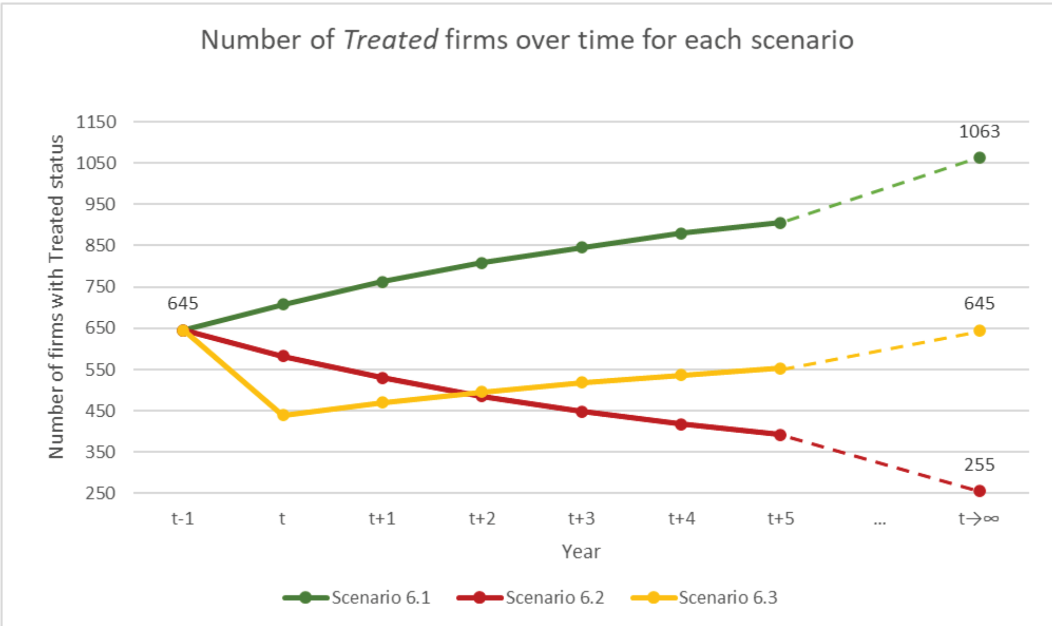

Figure 7 Change in number of *Treated* firms ($n_5 + n_6$) in each scenario

7 Conclusion

The framework detailed in this report offers a quantitative method for assessing the relationship between public investment into NPL's R&D activities and the number of firms accessing support over six-year periods. This is done by applying the principle of maximum entropy and modelling NPL's knowledge stock as a depreciating asset.

7.1 Findings

The marginal benefit of an additional year of support is shown to follow an exponential distribution, with its scale parameter directly influenced by NPL's accumulated knowledge stock. The results from illustrated examples demonstrate that variations in R&D funding have measurable effects on knowledge stock and firm participation and provide a useful tool for evaluating the long-term implications of investment decisions.

Scenario 1: 30% Increase in R&D Spend

New steady-state knowledge stock: £418.5 million (30% increase from £322*m*)
New steady state:

- *Pathway-to-Treated*: 30 additional firms (3% increase)
- *Close-to-Treated*: 84 additional firms (19% increase)
- ***Treated*: 418 additional firms (65% increase)**

Marginal benefit per instance of support increases across all engagement levels, but especially among *Treated* firms, due to higher knowledge stock.

Scenario 2: 30% Decrease in R&D Spend

New steady-state knowledge stock: £225.3 million (30% decrease from £322*m*)
New steady state:

- *Pathway-to-Treated*: 176 fewer firms (16% decrease)
- *Close-to-Treated*: 161 fewer firms (36% decrease)
- ***Treated*: 390 fewer firms (60% decrease)**

Marginal benefit per instance of support drops significantly, especially among *Treated* firms, reducing probability of continued engagement.

Scenario 3: One-Year Omission in R&D Spend

Temporary dip in knowledge stock: £273.6 million
Immediate impact:

- *Pathway-to-Treated*: 60 fewer firms (6% decrease)
- *Close-to-Treated*: 70 fewer firms (16% decrease)
- ***Treated*: 205 fewer firms (32% decrease)**

Gradual return to baseline engagement levels if funding resumes consistently.

7.2 Discussion

This framework assumes a steady-state distribution of firm engagement over time which aligns strongly with historical data from 2013-2023. It is important to note that while marginal cost of accessing support is treated as uniform across firms, in reality this may vary by sector, firm size, and type of engagement, e.g., a paid-for service and a collaboration on an innovation will likely incur different costs to a firm. Incorporating firm-level data on actual costs incurred during engagement in future iterations of this model could improve its accuracy. In the future, it could be useful to create variations of the model, i.e., with distinct parameters for the UK's sectoral groups. This is because different sectors interact with NPL differently owing to their technology readiness levels (TRL)[20]. It should also be mentioned that although this report estimates the value of an instance of support based on NPL's knowledge stock alone, external factors such as market conditions, policy revisions, or technological updates, can affect this variable.

[20] TRL stands for Technology Readiness Level, a nine-point scale developed by NASA to measure the maturity of a technology, from initial basic research to full-scale deployment.

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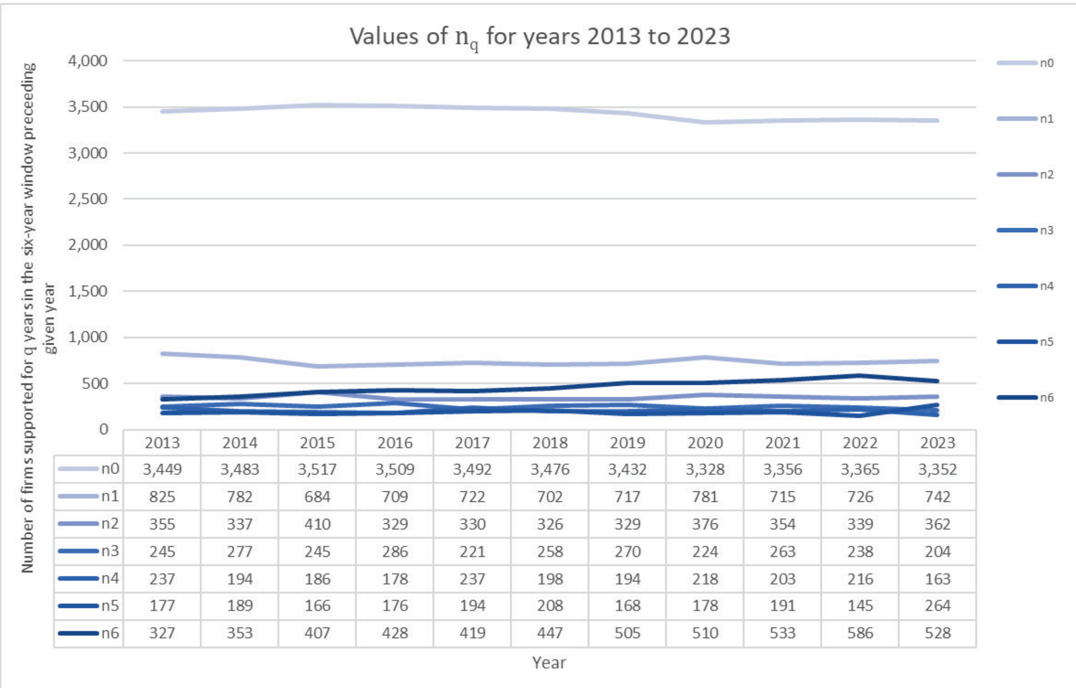
Annex

A. List of Assumptions

Below is a summary of all assumptions made in this report

1. **Fixed Population of Firms**: The total number of firms in the potential userbase, $\mathbb{N}$, is assumed to be fixed.
2. **Steady-State**: The distribution of user firms accessing $q \in \{0,1,2,\dots,6\}$ years of support during six-year long periods has reached a steady state based on historical data spanning 2013 to 2023.
3. **Firm Decision-Making**: A firm will choose to access an additional instance of support if the expected marginal benefit exceeds the marginal cost to the firm
4. **Fixed Marginal Cost**: The cost of accessing an instance of support is same for all firms and for all values of $q \in \{0,1,2,\dots,6\}$. Estimated using NPL's income from measurement services and applying the assumption that firms spend twice as much internally than what they pay NPL.
5. **Variable Marginal Benefit**: The marginal benefit accrued by firms from an instance of support is complementary to their properties and the value of q.
6. **Fixed Aggregate Investment**: The cumulative investment in assets, both tangible and intangible, is a conserved quantity which can redistributed among the user firms, but their aggregate value remains constant.
7. **Probability Distribution of Expected Marginal Benefit**: This is assumed to follow the exponential distribution based on the principal of maximum entropy.
8. **Knowledge Stock and Expected Marginal Benefit**: The expected marginal benefit from support is linearly related to NPL's knowledge stock via a parameter $\phi(q)$.

B. Values of n_q during 2013-2023



	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023
n0	3,449	3,483	3,517	3,509	3,492	3,476	3,432	3,328	3,356	3,365	3,352
n1	825	782	684	709	722	702	717	781	715	726	742
n2	355	337	410	329	330	326	329	376	354	339	362
n3	245	277	245	286	221	258	270	224	263	238	204
n4	237	194	186	178	237	198	194	218	203	216	163
n5	177	189	166	176	194	208	168	178	191	145	264
n6	327	353	407	428	419	447	505	510	533	586	528

Figure 8 Values of n_q for each year between 2013 to 2023 and the corresponding data table.

C. Marginal cost of accessing an instance of support from NPL

Presently, we do not have data regarding how much an organisation spends to access a year (instance) of support from NPL. In order to estimate this figure, the first step is to determine the average income received by NPL from a single firm. We have done this by dividing yearly totals of NPL's income from measurement services by yearly totals of number of firms supported given the existence of an invoice. We only consider income coming from firms directly and not funding provided by the government because, here, cost is representative of barrier to accessing support. While public funding enables NPL to support firms to a greater extent, it does not impact a firm's ability to solicit another year's support.

Year	Number of firms supported (given existence of invoice(s))	Total income from measurement services (in £ deflated)	Average income per firm (in £ deflated)
2013	182	8,119,049	44,610
2014	173	8,227,593	47,558
2015	175	7,273,895	41,565
2016	179	8,073,601	45,104
2017	139	7,974,157	57,368
2018	157	8,465,644	53,921
2019	173	8,948,573	51,726
2020	137	9,413,577	68,712
2021	146	10,074,360	69,002
2022	133	8,304,093	62,437
2023	122	8,985,502	73,652
AVERAGE			**55,969**

Table 6 Average income received by NPL from a firm per instance of support

According to the 2018 NMS Customer Survey, NMS users believe that for every £1,000 spent on innovation activities carried out by the NMS labs, a further £2,000 is spent by the firms on in-house innovation activities (King and Tellett, 2020)[21]. Applying this to the above figure, we get –

$$c = 3 \times £\ 55968.72 = £\ 167{,}906.15$$

[21] King, M; Tellett, G (2020) A customer survey for three of the core labs in the National Measurement System. Technical Report.

D. A continuous approach to Shannon entropy

Shannon entropy is for random variables taking discrete variables. The corresponding formula for continuous random variables is differential/continuous entropy. Similar to Equation 3.2, it measures the "uncertainty" or "information content" in a continuous distribution.

$$H(x) = \mathbb{E}[-\ln y'(x)] = -\int_x y'(x) \ln y'(x)\, dx$$

Here, x represents the valuation a firm assigns to one additional year of support from NPL, and $y'(x)$ is the pdf (probability distribution function) of x. Therefore, $\int_x y'(x) \ln y'(x)\, dx$ is an approximation of $\sum_{j=1}^{M} n_j . \ln(n_j)$ from Eq 3.2.

Euler-Lagrange Equation

Naturally, y is the cdf (cumulative distribution function) of x. $y(x)$ is a stationary point of $\int_a^b f(x,y,y')dx$ iff, $\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) = 0$

If f is independent of y then, $\frac{\partial f}{\partial y'} = \text{constant}$

The goal is to find the stationary point, provided the two constraints:

1. $\int y'dx = 1$ (i.e., integral over the pdf equals 1)
2. $\int xy'dx = \mu$ (fixed mean)

We find stationary points at $I(y)$ subject to a constraint by introducing Lagrange multipliers[13].

$$I = \int_a^b y'lny'dx + \lambda\overbrace{\left(\mu - \int xy'dx\right)}^{0}$$

$$I = \lambda\mu - \int_a^b (y'lny' + \lambda xy')xd$$

Let $f = y'lny' + \lambda xy'$

Since, y does not feature in f, the Euler equation reduces to: $\frac{\partial f}{\partial y'} = \text{constant}$

$$\Rightarrow \ln(y') + 1 + \lambda x = constant$$

$$\ln(y') = const - 1 - \lambda x$$

$$y' = Aexp(-\lambda x)$$

$$y' = \left[-\frac{A}{\lambda} exp(-\lambda x)\right]$$

$$y' = \frac{A}{\lambda}[1 - exp(-\lambda x)]\ for\ limits\ [0, x]$$

If $x \to \infty$, then $y \to 1$, which requires that $A = \lambda$.

Also, $\int xy'dx = \mu \Rightarrow \int x\lambda \exp(-\lambda x)\, dx = \mu \Rightarrow \lambda = 1/\mu$

Concept	Discrete approach	Continuous approach
Entropy	$-\sum_{j=1}^{M} n_j . \ln(n_j)$	$-\int_x y'(x) \ln y'(x)\, dx$
Constraints	$\mathcal{N} = \sum_{j=1}^{M} n_j , Y = \sum_{j=1}^{M} n_j\, y_j$	$\int y'dx = 1, \int xy'dx = \mu$
Result	$n_j \propto e^{-\lambda y_i}$	$y'(x) = \frac{1}{\mu} e^{-x/\mu}$

Table 7 Comparison summary table for discrete and continuous approach

E. Average values of n_q

q	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	$\bar{n}_q$	$\bar{N}(q)$
0	3,449	3,483	3,517	3,509	3,492	3,476	3,432	3,328	3,356	3,365	3,352	3,433	5,615
1	825	782	684	709	722	702	717	781	715	726	742	737	2,182
2	355	337	410	329	330	326	329	376	354	339	362	350	1,446
3	245	277	245	286	221	258	270	224	263	238	204	248	1,096
4	237	194	186	178	237	198	194	218	203	216	163	202	848
5	177	189	166	176	194	208	168	178	191	145	264	187	645
6	327	353	407	428	419	447	505	510	533	586	528	458	458

Table 8 Average values of n_q and $N(q)$ based on historical data (2013-2023)

F. Illustrative examples: background calculations

Scenario 6.1

Year	t	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t\to\infty$
$R\ (in\ £)$	62,774,279	62,774,279	62,774,279	62,774,279	62,774,279	62,774,279	**62,774,279**
$K\ (in\ £)$	336,405,752	348,719,168	359,185,572	368,082,015	375,643,992	382,071,672	**418,495,193**
$\varphi_1 K (in\ £)$	185,635	192,430	198,205	203,114	207,287	210,834	**230,933**
$\varphi_2 K (in\ £)$	425,741	441,325	454,571	465,830	475,400	483,534	**529,631**
$\varphi_3 K (in\ £)$	632,626	655,782	675,464	692,194	706,415	718,503	**786,999**
$\varphi_4 K (in\ £)$	683,239	708,248	729,505	747,574	762,932	775,986	**849,963**
$\varphi_5 K (in\ £)$	644,014	667,587	687,624	704,655	719,132	731,437	**801,166**
$\varphi_6 K (in\ £)$	512,475	531,234	547,178	560,731	572,250	582,042	**637,529**
N_1/N_0	0.4047	0.4179	0.4286	0.4375	0.4449	0.4510	**0.4833**
N_2/N_1	0.6741	0.6835	0.6912	0.6974	0.7024	0.7066	**0.7283**
N_3/N_2	0.7669	0.7741	0.7799	0.7846	0.7884	0.7916	**0.8079**
N_4/N_3	0.7821	0.7889	0.7944	0.7988	0.8025	0.8054	**0.8207**
N_5/N_4	0.7705	0.7776	0.7833	0.7880	0.7918	0.7949	**0.8109**
N_6/N_5	0.7206	0.7290	0.7358	0.7412	0.7457	0.7494	**0.7685**
N_1	2273	2346	2407	2457	2498	2532	**2714**
N_2	1532	1604	1664	1713	1755	1789	**1977**
N_3	1175	1242	1298	1344	1384	1416	**1597**
N_4	919	980	1031	1074	1111	1140	**1311**
N_5	708	762	808	846	880	906	**1063**
N_6	510	556	594	627	656	679	**817**
n_1	741	742	743	744	743	743	**737**
n_2	357	362	366	369	371	373	**380**
n_3	256	262	267	270	273	276	**286**
n_4	211	218	223	228	231	234	**248**
n_5	198	206	214	219	224	227	**246**
n_6	510	556	594	627	656	679	**817**

Table 9 Calculations to determine number of firms with q instances of support, n_q, based on the increase in yearly R&D spend

Scenario 6.2

Year	t	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t\to\infty$
R (in £)	33,801,535	33,801,535	33,801,535	33,801,535	33,801,535	33,801,535	**33,801,535**
K (in £)	307,433,007	295,119,591	284,653,187	275,756,744	268,194,767	261,767,087	**225,343,566**
$\varphi_1 K$(in £)	169,647	162,852	157,077	152,168	147,995	144,448	**124,349**
$\varphi_2 K$(in £)	389,075	373,491	360,246	348,987	339,416	331,282	**285,186**
$\varphi_3 K$(in £)	578,141	554,985	535,303	518,573	504,352	492,265	**423,769**
$\varphi_4 K$(in £)	624,396	599,387	578,130	560,061	544,703	531,648	**457,672**
$\varphi_5 K$(in £)	588,549	564,976	544,939	527,908	513,431	501,126	**431,397**
$\varphi_6 K$(in £)	468,339	449,581	433,636	420,084	408,564	398,772	**343,285**
N_1/N_0	0.3717	0.3566	0.3434	0.3317	0.3216	0.3127	**0.2592**
N_2/N_1	0.6495	0.6379	0.6275	0.6181	0.6098	0.6024	**0.5550**
N_3/N_2	0.7479	0.7389	0.7308	0.7234	0.7168	0.7110	**0.6729**
N_4/N_3	0.7642	0.7557	0.7479	0.7410	0.7347	0.7292	**0.6929**
N_5/N_4	0.7518	0.7429	0.7348	0.7276	0.7211	0.7153	**0.6776**
N_6/N_5	0.6987	0.6883	0.6790	0.6705	0.6630	0.6564	**0.6132**
N_1	2087	2003	1928	1863	1806	1756	**1455**
N_2	1356	1278	1210	1151	1101	1058	**808**
N_3	1014	944	884	833	789	752	**544**
N_4	775	713	661	617	580	548	**377**
N_5	583	530	486	449	418	392	**255**
N_6	407	365	330	301	277	257	**156**
n_1	731	725	718	712	705	698	**647**
n_2	342	334	326	318	312	306	**264**
n_3	239	231	223	216	209	204	**167**
n_4	192	183	175	168	162	156	**122**
n_5	176	165	156	148	141	135	**99**
n_6	407	365	330	301	277	257	**156**

Table 10 Calculations to determine number of firms with q instances of support, n_q, based on the decrease in yearly R&D spend

Scenario 6.3

Year	t	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t\to\infty$
$R\ (in\ £)$	-	48,287,907	48,287,907	48,287,907	48,287,907	48,287,907	**48,287,907**
$K\ (in\ £)$	273,631,473	280,874,659	287,031,367	292,264,569	296,712,790	300,493,779	**321,919,379**
$\varphi_1 K(in\ £)$	150,995	154,992	158,389	161,277	163,731	165,818	**177,641**
$\varphi_2 K(in\ £)$	346,297	355,464	363,255	369,878	375,508	380,293	**407,408**
$\varphi_3 K(in\ £)$	514,576	528,197	539,775	549,616	557,982	565,092	**605,384**
$\varphi_4 K(in\ £)$	555,745	570,456	582,960	593,589	602,623	610,302	**653,817**
$\varphi_5 K(in\ £)$	523,839	537,706	549,492	559,510	568,026	575,264	**616,282**
$\varphi_6 K(in\ £)$	416,846	427,880	437,259	445,231	452,008	457,768	**490,407**
N_1/N_0	0.3289	0.3385	0.3464	0.3531	0.3586	0.3633	**0.3886**
N_2/N_1	0.6158	0.6235	0.6299	0.6351	0.6395	0.6431	**0.6622**
N_3/N_2	0.7216	0.7277	0.7327	0.7368	0.7401	0.7429	**0.7578**
N_4/N_3	0.7392	0.7450	0.7497	0.7536	0.7568	0.7595	**0.7735**
N_5/N_4	0.7258	0.7318	0.7367	0.7407	0.7441	0.7469	**0.7615**
N_6/N_5	0.6684	0.6754	0.6811	0.6858	0.6897	0.6930	**0.7101**
N_1	1847	1901	1945	1982	2014	2040	**2182**
N_2	1137	1185	1225	1259	1288	1312	**1445**
N_3	820	862	898	928	953	975	**1095**
N_4	606	642	673	699	721	740	**847**
N_5	440	470	496	518	536	553	**645**
N_6	294	317	338	355	370	383	**458**
n_1	710	716	720	723	726	728	**737**
n_2	317	323	327	331	335	337	**350**
n_3	214	220	225	229	232	235	**248**
n_4	166	172	177	181	185	187	**202**
n_5	146	153	158	163	166	170	**187**
n_6	294	317	338	355	370	383	**458**

Table 11 Calculations to determine number of firms with q instances of support, n_q, based on a one-off omission in R&D funding

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