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A MACROECONOMIC MODEL WITH IMPERFECT PRODUCTION PROCESSES AND CONFORMANCE TESTING

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ABSTRACT

This study shows how the contribution of conformance testing can be included in a macroeconomic model where the production process sometimes generates defective outputs. Specifically, this study adapts the Solow model by making the economy's Total Factor Productivity (TFP) a function of the effort committed by engineers to conformance testing. Although, this study takes a theory-based approach, the resulting model can be operationalised using plausible estimates of the key parameters. The aim is thereby to provide theoretical underpinnings for the conformance testing aspects of the national quality infrastructure. Furthermore, the model can be used for a benefit-cost analysis of the National Measurement System (NMS) programme, which is responsible for maintaining the UK's measurement infrastructure. Finally, this model will really come into its own once there are updated estimates of the core parameters from a forthcoming NMS survey.

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1 EXECUTIVE SUMMARY

This report takes a theory-based approach to bring together insights from earlier reports and integrate them into a new macroeconomic model for the measurement-related aspects of the national quality infrastructure. The three earlier reports are:

- King, M. & Renedo, E. (2020). Estimating the Price Elasticity of Demand for NPL's Services. NPL Report. IEA 6.
- Fennelly, C. (2021). Quantifying Measurement Activity in the UK. NPL Report. IEA7.
- King, M. & Nayak, S. (2023). An Economic Model for the Value Attributable to High-Quality Calibrations by Reducing Mistakes in Conformance Testing. NPL Report. IEA 19.

Insights from each of these reports can be usefully brought together by considering an economy in which production processes sometimes malfunction but where reliability is maintained through regular conformance testing (CT). That is, our report establishes a macroeconomic model in which production processes sometimes generate defective outputs, but where such incidents can be detected, and then corrected, through an economy-wide system of quality control. Lastly, the earlier reports provide estimates for some of the key parameters in this new macroeconomic model, and so enable our model to generate quantitative results.

1.1 THE CONCEPTUAL FRAMEWORK

We develop a macroeconomic model for the yearly output from the "real economy", defined as private sector organisations excluding the financial sector. Specifically, this study extends a canonical model from macroeconomics (called the Solow model) by modifying its setup to allow for the possibility of defective outputs due to malfunctions within the production process.

To stop the reliability of the production process from degenerating, a small proportion of the economy's output goes towards paying for engineers to undertake conformance testing. Output from machines that fail such tests is scrapped, and only the output from machines that pass these tests is allowed to enter the supply chain. Hence, the CT engineers search through the capital stock looking for the malfunctioning machines that need to be reset.

Consequently, this study derives a model for the costs and benefits of an economy-wide conformance testing regime; and it can be seen as an attempt at developing a quantitative "systems" model for the national quality infrastructure. However, as this study has been developed by economists at NPL, it mostly focuses on the measurement activities associated with conformance testing. Hence, it's acknowledged, at the outset, that this study undoubtedly fails to account for other benefits from the system that lack a direct connection to conformance testing. Hence, yet further models will be needed to account for other benefits from the national quality infrastructure.

1.2 SETTING UP THE MODEL

Per capita output (labour productivity) is defined as the Gross Value Added (GVA) per worker, of which the principal determinants are:

- The "capital intensity" of the economy, defined as the amount of capital (machinery and equipment) per worker.
- The "reliability" of the production process, defined as the proportion of the capital stock that's working correctly.

Suppose that a capital item is either in a "good" state (working correctly) or a "bad" state (malfunctioning). The "good" machines are producing outputs that conform to specification, whereas the "bad" machines are producing defective outputs that must be scrapped.

Moreover, it's possible for machines to switch from "good" to "bad", and the likelihood of such transitions is determined by the "transition rate". (The "transition rate" is the proportion of "good" machines that go "bad" each year.) However, the malfunctions are not obvious to the workers, meaning that they can't tell, just by eye, when outputs are defective. Hence, specialist engineers are tasked with searching through the capital stock to find the malfunctioning machines that need to be reset. These conformance testing activities are characterised by the following parameters:

- The "regret rate" is the likelihood of accidentally scrapping usable output due to type-1 errors (false positives) in the testing process.
- The "detection rate" is the likelihood that a malfunctioning machine is correctly reset so that it once again produces usable output.

Suppose that each engineer is responsible for supervising a portion of the capital stock (measured in millions of pounds of capital). The "pace-of-testing" is the maximum amount of capital that an engineer can comb through in a year whilst still ensuring that the tests are reliable. (Note that the "pace-of-testing" is a measure of the productivity of the CT engineers.) Together, the "portion-size" and the "pace-of-testing" determine the frequency of inspections.

1.3 THE STEADY-STATE EQUILIBRIUM

The model yields a system of differential equations: one for the evolution of the capital intensity and another for the reliability of production. The dynamics of the system can be pictured in a two-dimensional phase diagram, where the axes are the economy's capital intensity (vertical axis) and the reliability of production (horizontal axis). Moreover, this phase diagram can be used to find an equilibrium in which both variables remain constant (fixed at their steady state values). The formulae for the steady-state values of these variables provide the mathematics behind a theory-of-change, which explains how changes in the basic parameters of the model affect economic outcomes:

- Labour productivity is positively affected by the efficiency of production, meaning that workers' wages would rise if conformance tests became better at discriminating between "good" and "bad" outputs. Specifically, the efficiency of production depends on the chance that perfectly viable output falls prey to type-1 errors ("false positives").
- The equilibrium level of the economy's capital intensity increases, because having more capital per worker increases labour productivity. This causes a rise in per capita income, meaning that citizens become more prosperous.
- With a fixed savings rate, a rise in per capita income leads to a larger pool of savings, which can then be used by businesses to fund their investments in new capital equipment.

Therefore, capital intensity depends on the prevalence of type-1 errors ("false positives") through its connection to the efficiency of production. Most importantly, if the prevalence of type-1 errors rises, then the efficiency of production decreases, leading to a decline in capital intensity.

Next, in the short term, one would expect the rental rate (corresponding to the marginal product of capital) and capital intensity to move in opposite directions. However, in the long run, the positive effect (on the marginal product of capital), from an improvement in the efficiency of production, almost exactly offsets the negative effect on the rental rate from an increase in capital intensity (through an increase in the supply of capital). It follows that the equilibrium rental rate will hardly change even when engineers get better at finding malfunctioning machines. In other words, an improvement in the efficiency of production increases the demand for capital but, in equilibrium, the price of capital remains almost unchanged. Such results are one of the benefits of using general equilibrium models rather than partial equilibrium models.

Lastly, in the steady state, the net rental rate (rents minus the cost of engineers) is proportional to the gross investment rate. This is a version of Piketty's famous formula, as espoused in his book: '*Capital in the Twenty-First Century*.' Next, it can be shown that the economy's net revenue (revenue minus the cost of engineers) is proportional to the level of gross investment. So, in equilibrium, society's per capita consumption ("prosperity") is an increasing function of the economy's capital intensity. Hence, anything that increases the capital intensity (such as, engineers getting better at discriminating between "good" and "bad" outputs) also increases peoples' living standards, which feeds back on capital intensity by increasing the flow of savings used for investment.

1.4 THE OPTIMAL INSPECTION FREQUENCY

Conducting more inspections would entail higher costs because more resources would be allocated to conformance testing and a higher fraction of viable output would fall prey to type-1 errors in the testing process. However, more inspections would also increase the detection rate, which would then boost the reliability of the economy's production processes. This trade-off between the costs and benefits of extra inspections implies that there is an optimal inspection frequency. (Ultimately, the optimal inspection frequency is that which sustains the highest possible equilibrium capital intensity, because it's the capital intensity that determines the prosperity of citizens.)

The first-order condition for the optimal inspection frequency can be written as:

$$\frac{(\text{Spend on CT})}{(\text{Frequency})} = \frac{[1 - \Pr\{\text{type 2}\}] \times (\text{Rebate Rate}) \times (\text{GVA})}{(\text{Gross Investment Rate}) + (\text{Detection Rate})} - \Pr\{\text{type 1}\} \times (\text{GVA})$$

Where, the various quantities that feature in this expression are as follows:

- 'Spend on CT' is the yearly cost of employing CT engineers to conduct inspections.
- 'Frequency' is the average number of times a batch of outputs will be inspected before it enters the supply chain.
- 'GVA' is the Gross Value Added of the UK's real economy at factor cost (that is, it's the economy's yearly output).
- 'Gross Investment Rate' is the proportion of the capital stock that needs to be replaced each year to maintain the steady state.
- 'Rebate Rate' is the proportion of output that is sent back to suppliers by customers when they realise it's defective.
- 'Detection Rate' is the likelihood that a batch of defective output is picked up by the conformance tests before it enters the supply chain.
- $\Pr\{\text{type 1}\}$ is the likelihood of a type-1 error (or false positive); and $\Pr\{\text{type 2}\}$ is the likelihood of a type-2 error (or false negative). This means that the statistical power of the test is $1 - \Pr\{\text{type 2}\}$.

The mathematical form of this first-order condition indicates that the various terms represent distinct positive and negative influences on equilibrium capital intensity, each emanating from a small variation in the frequency of inspections. Moreover, the elements in this expression clearly correspond to some economically meaningful quantities:

- The lefthand side of this expression gives us an equation for the cost of one complete sweep of the capital stock. In other words, this quotient gives us the unit cost for a full round of inspections.
- The righthand side of this expression represents the net benefit from another round of inspections; with the positive term representing the additional output from more reliable production processes, and the negative term representing the loss from a little more of the viable output falling prey to type-1 errors in the testing process.

Hence, the equation above takes the form of a "*marginal cost equals marginal benefit*" type of optimality condition. That is, the lefthand side is the marginal cost of a round of inspections and the righthand side relates to the benefit from a round of inspections. In other words, optimality requires that the marginal cost of another round of inspections equals the marginal benefit.

1.5 HEADLINE RESULTS

The pace at which an engineer can inspect the machines under their supervision depends on the quality of the infra-technology providing the technical basis for standards. Much of this infra-technology is grounded in the science of metrology, which constitutes a kind of "public good" that is developed and maintained by the specialist laboratories funded through the National Measurement System (NMS) programme. Moreover, the NMS labs maintain and update the primary standards that underpin a distributed system for the certification of calibrations, and for ensuring their comparability to corresponding standards around the world. Calibrations traceable to these primary standards are delivered to more than 74,000 businesses via a network of calibration labs distributed across the UK. (Since a top-tier calibration lab can supply calibration services to a second-tier calibration lab, this estimate of 74,000 businesses represents just the first tier of fanout across the economy.)

The benefit-cost analysis shows that cuts in NMS funding would lead to a loss of economic benefits. We use the lower bound (Scenario 2) as the most reasonable estimate of what would be lost without the NMS labs. The upper bound (Scenario 1) gives an estimate of what would be put at risk without the NMS labs. So, the lower bound is what the UK would surely lose, and the upper bound for the value of what would be put in jeopardy.

Before outlining the impact of the NMS, a few key things must be noted. Firstly, a marginal cut to the NMS would be less significant per pound of saving than if the programme were scrapped in its entirety. In essence, a marginal cut to benefits is less harmful than an average cut in benefits. Secondly, if funding for the UK's measurement infrastructure was cut in its entirety, then the programme's other benefits mechanisms (research, innovation, knowledge transfer) would also cease to operate. Finally, there are two kinds of possible loss depending on the scale of the cut:

- If the NMS was cut in its entirety, the average return on public funding should be used to get an estimate of the economic damage.
- If the NMS lost a proportion of its funding but continued as a programme, then the marginal return on public funding should be used to get an estimate of economic losses.

This economic analysis shows that if the UK stopped funding the NMS labs but came to an arrangement with a foreign National Measurement Institute (NMI), such as, VSL in the Netherlands, then we would see an average social loss to the economy of £8.79 for each £1.00 saved by the government due to no longer funding the NMS.

- "Basic calibrations" have a Test Accuracy Ratio (TAR) of around 1:4. In contrast, using "precise calibrations", reduces the Relative Standard Deviation (RSD) of measurements by about 3%.
- Amongst the many thousands of businesses that directly (or indirectly) depended on the NMS labs, the extra cost of having to go abroad to access "precise calibrations" leads to a 31% fall in the use of calibrations traceable to highly accurate national standards.
- A decline in the use of "precise calibrations", reduces the effectiveness of the businesses' conformance testing activities, because measurements become less reliable and more prone to "false positives". Consequently, we would expect the amount spent by businesses on conformance testing to fall by 1.1%. Given that the

UK currently spends around £28 billion on conformance testing, this cut in spending yields a saving of £317 million for businesses.

- However, the decrease in conformance testing also causes a drop in the marginal product of capital, which leads to a 0.1% decline in equilibrium capital intensity. (This assumes that the equilibrium cost-of-capital is fixed by parameters that remain unchanged.) In equilibrium, savings must equal investment, which means that labour productivity must be proportional to capital intensity. From this, we find that the decline in output per worker results in a loss to the economy of £1.1 billion in GVA. (This is a lower bound for the loss to the economy, given that it excludes the benefits coming from supporting innovation amongst businesses.) Lastly, since the government would save £80 million from scrapping the NMS, the end result is a net economic loss to the UK of £700 million in GVA.

Of more relevance to a government spending review is the possibility of a cut some of the funding for the NMS programme. That is, the NMS could experience a cut in funding, forcing it to withdraw from various areas of measurement in proportion to the size of the cut. For context, the NMS currently covers about 75% of the Core Measurement Capabilities as outlined by BIPM's database.[1] If the NMS labs scaled back their offering, then businesses requiring high accuracy calibrations in the “mothballed” areas would have to send their instruments to a foreign National Measurement Institute (NMI). The analysis in this report shows that this would lead to a marginal social loss of £5.46 per £1.00 saved through cuts to the programme. Note that this estimate of the marginal return includes a discount factor for a presumed 6-year delay in the effects being felt in the economy. This is due to the cuts falling upon the development of the measurement infrastructure, not the projects maintaining existing measurement infrastructure, which would still be safeguarded.

Lastly, our analysis shows how the model’s parameters determine the behaviour of the system, where the numerical analysis uses values for the period 2015 to 2019. There are two main reasons for this is choice of time period: Firstly, the model is based on a series of long run equilibrium relationships that would have been disrupted by the Covid pandemic. Secondly, the previous empirical studies span this period, and so give us a set of consistent parameter values. However, one consequence of this choice of time period is that more work is needed to establish updated values for the model’s parameters, which will become the subject of future empirical studies. Nonetheless, the ability to identify the system’s key parameters, and then combine them in a consistent model, brings us a lot closer to a full quantification of the costs and benefits. Indeed, such parameters may ultimately form the basis of metrics to track changes in the performance of the system. That is, it might someday be possible to monitor the performance of the system in much the same way as crime statistics are used to monitor the police and justice system.

2 INTRODUCTION

Production processes sometimes generate defective outputs, but such incidents can be detected and corrected through an economy-wide system of quality control, known as the national quality infrastructure.

Before discussing the details of the approach taken in this study, it’s useful to begin by introducing the concept of a national quality infrastructure, along with an explanation of why a model for the benefits of this infrastructure is required.

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[1] The Bureau International des Poids et Mesures (BIPM) is the international organization through which Member States work together on matters related to metrology.

2.1 THE NATIONAL QUALITY INFRASTRUCTURE

A measurement is only useful in so far as it is both reproducible and comparable; and so, to underpin this reliability there exists a series of top-level measurement laboratories funded by the Department for Science, Innovation, and Technology (DSIT). The National Measurement System (NMS) begins with these top-level laboratories (NPL, NML, NEL, NGML, NIBSC)[2] but it extends out to include a community of accredited labs providing certified reference materials or calibrations that are traceable back to working standards held by the top-level laboratories. These accredited providers of measurement services are essential to the fan-out of traceable calibrations across the economy, thereby ensuring the comparability of measurements made in the many different organisations even though they don't directly use the top-level NMS laboratories. By this means, a distributed system of traceable calibrations helps to ensure that organisations can have confidence in the measurements they make, or that are made on their behalf. Finally, at its core, the NMS is about ensuring that measurements made in the UK are consistent with the global common system of measurement units: the International System of Units – the SI (Système international d'unités). Consequently, a regular series of key-comparison exercises are conducted with counterparts in other countries to maintain the Mutual Recognition Arrangements (MRAs) that negate technical barriers to trade that would otherwise impede international trade.

It is important to recognise that the National Measurement System (NMS) fits into a broader system of standards and accreditation. Standards codify what a community of experts has come to regard as the best available method for achieving an outcome that meets a given specification. Hence, much of standardisation is concerned with the codification of the tacit knowledge that has been acquired through long experience but must then be written down so that it can form the basis of learnable routines. Next, accreditation provides assurance to buyers that a supplier is following best practice, as laid down in the relevant standards, and so underpins confidence in the certification process (e.g., CE marking). The justification for wanting to encourage the use of standards and accreditation is that the adoption of best practice will reduce the frequency of errors. Lastly, changes to technology, markets, and regulations require a continuous flow of updates (both retirements and additions) to the stock of standards, as well as to the enabling infra-technology. That is, the infrastructure must continue to evolve so that it remains relevant to what's happening within the economy and society.

This interconnected system is referred to as the national quality infrastructure and, along with the NMS, it includes both the British Standards Institute (BSI) and the United Kingdom Accreditation Service (UKAS). This technical and legal infrastructure ensures buyers (businesses and consumers) can have justifiable confidence in the goods and services they purchase. Consequently, when this system is working well, buyers take it for granted that certificated products are of dependable quality, conform to the prescribed specification, and meet any regulatory requirements.

Standards play a role in resolving compatibility issues that can sometimes create difficulties for the evolution of multiproduct systems. That is, standards help to set the characteristics of the interfaces connecting distinct components of a multiproduct system. These interface standards are important for ensuring that the components, being developed by different suppliers, are compatible with one another so that they can be assembled in a way that allows the resulting system to function optimally.

This confidence is critical to ensuring that transaction costs are kept to a minimum so that goods can be bought and sold in markets where both sides of the transaction truly understand the quantity and quality of what is being exchanged. Without such confidence, buyers would be compelled to conduct their own tests, with attendant costs and delays that

[2] National Physical Lab (NPL); National Measurement Lab (NML); National Engineering Lab (NEL); National Gear Metrology Lab (NGML); National Institute for Biological Standards and Control (NIBSC).

eat into the gains from business-to-business transactions. Reducing transaction costs, and thereby growing the volume of business-to-business transactions, leads companies to buy in commoditised components and outsource some of their more routine operations; thereby shrinking the scope and complexity of what needs to be managed in-house. This helps to reduce the boundaries of the firm, and so underpins the gains in productivity coming from economies-of-scale and specialisation.

Similarly, standards and accreditation can reduce compliance costs for companies operating in industries whose products or processes are regulated because of potential harm to employees, consumers, or the environment. That is, standards can be developed to offer companies a clear route to demonstrating compliance with regulations, thereby, de-risking some of their operations. Hence, this technical and legal infrastructure also includes the Office for Product Safety and Standards (OPSS): a unit within the Civil Service that's responsible for providing the surveillance that underpins the proper functioning of product markets. One aspect of this surveillance concerns checking that businesses comply with any relevant Weights & Measures Legislation (Legal Metrology).

2.2 ROUTES TO IMPACT FOR THE NMS LABORATORIES

NPL's reliance on funding from the NMS programme means that it needs to be able to justify its activities in terms of the benefits that it creates for the UK. Since government officials are responsible for ensuring that taxpayers' money is used in the best interests of citizens, the case for continuing to fund a given institution begins by taking officials through a logic model that explains how this institution generates a certain set of benefits for society. In NPL's case, the well-established logic model for the National Measurement System (NMS) was built around the following four impact mechanisms:

1. Research and Development – working with research-based organisations to advance fundamental metrology and the broader science of measurement.
2. Direct Support for Innovation – collaborating with businesses and public sector organisations to support their innovation projects.
3. Traceability and Standards – supplying the high-level calibrations and reference materials that underpin the comparability and reproducibility of measurements.
4. Knowledge Transfer – providing training and consultancy that helps businesses and other organisations to improve their in-house measurement capabilities.

Given that mechanisms 1 and 2 are so closely connected, they can be brought together under a broader channel called "*enabling technological change*". Similarly, as mechanisms 3 and 4 are interrelated, they can come together under a channel called "*delivering the measurement infrastructure*".

Since the first channel (*enabling technological change*) is concerned with innovation, it fits neatly into a well-established subfield of economics, where the fundamental thinking has already been done.[3] In contrast, the second channel (*delivering the measurement infrastructure*) is concerned with what might be seen as the niche topic of infra-technologies (e.g., metrology), whose importance seems to have been somewhat overlooked by most academic economists.[4] Consequently, this study is devoted to building a theoretical model for this important, but little noticed, piece of national infrastructure.

[3] Growth accounting and endogenous growth models are well-established parts of economics thanks to early work by Kenneth Arrow and Paul Romer, both Nobel prize-winning economists. Furthermore, there are a range of established field journals for the economics of innovation and science policy.

[4] There have been foundational contributions from economists, such as Greg Tassey, Peter Swann, Knut Blind, and Richard Hawkins, amongst others. However, the topic is too niche for there to be dedicated journals for the economics of infra-technologies and standardisation. The closest there is to such a field journal is *Information Economics and Policy*. Lastly, much of the work on standards is a sub-element within other well-established areas of economics, such as the economics of innovation or the economics of trade.

In the absence of a quantitative economic model for infra-technologies, the econometric evidence for the NMS is based on the value of attributable innovations amongst the group of businesses that are regular users of the NMS laboratories. Consequently, the existing evidence fails to account for the kind of infrastructural benefits that are unconnected to innovation. Specifically, the existing evidence misses the day-to-day benefit of sustaining the effectiveness of production processes whose reliability is underpinned by the work of UKAS-accredited laboratories providing calibrations that are traceable to national standards maintained by the NMS labs. This constitutes a significant limitation to our ability to account for routine benefits that flow from the NMS programme. This gap in our evidence is unfortunate because it is the benefits coming from the measurement infrastructure that truly distinguishes the NMS programme from all the other research and technology programmes.

2.3 RATIONALE AND MOTIVATION FOR THIS STUDY

The origins of metrology and standardisation stretch back to the 19th-century industrial policy; meaning that the decision to establish NPL dates from a time before economics had cemented its place as the dominant paradigm for public policy. And, because NPL predates the current paradigm it has somewhat avoided the fundamental economic questions encountered by new programmes. Furthermore, when challenged, during HM Treasury's comprehensive spending reviews to demonstrate a tangible benefit to the UK, NPL has long been able to point towards robust econometric evidence for enhancing innovation among the group of businesses that regularly use its services. This will always remain an important and legitimate part of the argument for continuing to fund NPL, but it isn't a complete account of what it does for the UK. Moreover, because so much of NPL's infrastructural benefit is difficult to quantify this aspect of its work has not featured prominently in previous spending reviews.[5] The consequence of overlooking the benefits coming from the infrastructural side of the NMS is that, although, NPL can demonstrate a positive net-benefit, solely from its role in supporting innovation, the institution doesn't score as highly as it could in the department's value-for-money rankings.

For the reasons outlined above, a new macroeconomic model is needed to provide a better account of the "system" aspects of the NMS. Hence, this study sets out a new approach to modelling the system's benefits that places the UK's measurement infrastructure within this broader quality-system. More specifically, this study derives a model for the costs and benefits of the national quality infrastructure that is grounded in conventional macroeconomics, with the aim of providing a theoretical account of the system that is rooted in macroeconomic thinking. The analysis contained in this study shows how the model's parameters determine the behaviour of the system.

The ability to identify the key parameters, and to then combine them in a consistent model, brings us much closer towards our goal of arriving at a complete quantification of the costs and benefits. Indeed, such parameters may ultimately give an economic basis to metrics that would then help us to track changes in the performance of the system. By this means, it might be possible to monitor the performance of the system in much the same way as crime statistics are used to monitor the performance of a police force.

This study should be seen as an attempt at developing a "systems" model. However, establishing updated values for these parameters will have to become the subject of yet further empirical studies or industry surveys. Moreover, as it has been developed by economists at NPL, it comes with an inherent focus on the conformance testing aspects of the national quality infrastructure. No doubt, such a model fails to account for benefits lacking a connection to conformance testing, such as, the lowering of barriers to international trade.

[5] Some studies have estimated the scale of spending on measurement across the economy. A recent example of such a study is: Fennelly, C (2021) Quantifying measurement activity in the UK. NPL Report. IEA 7. (This study is available at: http://eprintspublications.npl.co.uk/9064/.) Studies of this nature have found that the UK's annual spending on measurement runs into the tens of billions. However, until now, there has been no way to work back from these values to an estimate of the extra benefit that is attributable to the UK's measurement infrastructure.

Hence, it is recognised, from the outset, that the model developed in this report still isn't the full story, and so yet further models will be needed to account for aspects of the quality-system that cannot be seen through a window of conformance testing.

Lastly, this report is structured as follows:

Part 1: Chapter 3 gives a brief literature review relating to the economics of standards and infra-technologies. Chapters 4 and 5 explain the conceptual framework and backstory to the model. Chapter 6 provides headline statistics for the economy, that can be identified with some of the parameters of the model. Chapters 7, 8, and 9 set up and solve the Solow model. These chapters also explain the connection between the Solow model and the "laws of capitalism" espoused by Thomas Piketty.

Part 2: Chapters 10 - 14 introduce new elements into the model to account for the effects of imperfect production processes and conformance testing activities. Chapters 15 and 16 derive differential equations (state equations) for the capital stock and reliability of the production process. This yields a system of two coupled differential equations for the dynamics of the economy. Chapter 17 analyses the behaviour of the system and characterise the equilibrium (steady state). This analysis combines a graphical approach using phase diagrams with a more mathematical approach based on the properties of the system's Jacobian matrix – details of which are given in Annex B.

Part 3: Chapters 18 - 21 make inferences about the influence of the parameters on the system's equilibrium. These chapters use techniques from the field of comparative statics to explore the effect of changes in the frequency of inspections on equilibrium capital intensity. Chapters 22 and 23 derive a condition for the optimal inspection frequency, and thereby endogenize the amount spent by businesses on conformance testing. Chapter 24 uses a numerical method to solve a set of simultaneous equations; and thereby finds values for the model's unknown parameters.

Part 4: Chapters 25 and 26 explore the effect of calibration-related measurement uncertainty on the Relative Standard Deviation (RSD) of measurements used for conformance testing. Specifically, they introduce the concepts of "basic calibration" and "precise calibration". Chapter 27 explores what would happen if the UK were to take traceability from a foreign NMI, such as, VSL in the Netherlands. It shows that the use of "precise calibrations" would drop by around 31% due to cost of having to go abroad to get calibrations traceable to highly reliable national standards. Chapter 28 introduces two scenarios for what might happen if the NMS labs were defunded.

Part 5: Chapters 29 – 31 explore the effect of changes in the accuracy of measurements on the likelihood of type-1 errors (false positives) in the conformance testing process. Chapter 32 uses results from the two scenarios to conduct a benefit-cost analysis for the NMS programme.

The final chapters outline ideas for further work and summarises the findings of this study. A separate addendum to this report contains three annexes: Annex A gives a detailed list of data sources; Annex B uses the Hartman-Grobman theorem to prove that the system has a unique stable equilibrium; and Annex C provides a graphical analysis of the dynamics of system in a two-dimensional phase space.

3 LITERATURE REVIEW

As discussed in the previous section, there is little in the existing economics literature that builds a quantitative economic model for infra-technologies. Nonetheless, at least three economists[6] (Swann, Tassey, Blind) have made significant contributions towards the development of an intellectual framework for understanding the value of measurement.

Routine measurements provide information about specific things at a particular time, and confidence in this information is enhanced by using precisely calibrated instruments. Generally, a simple way to judge the economic value of a service is by observing the willingness-to-pay of its users. However, since the benefits of metrology are freely embedded into almost every aspect of the economy, spillovers insert a wedge between individual private benefits and the wider societal benefits. In other words, primary standards are close to being a kind of public good that can be accessed almost for free by linking to the chain of traceable calibrations that are anchored to such standards. The absence of copyright protection for calibrations means that they can be copied very cheaply and the benefits passed on to other users without any payments being collected by the NMS labs.

This makes putting a number on the true value of a national measurement infrastructure a challenging task. Nonetheless, a few studies have attempted to quantify the spending on measurement activity across an economy as a share of its GDP to arrive at some kind of a lower bound for the value of measurement (Huntoon 1967; Paulson 1977; Don Vito 1984; Williams 2002).

Tassey (1982) introduces the idea that measurement standards should really be seen as a public “infratechnology”: technical tools, in the form of freely available information goods, that enable the development, production and use of other technologies. That is, infratechnologies provide techniques that can be widely applied across a range of sectors to enable further innovation. Tassey (1986) analyses the economic role of the National Bureau of Standards within the context of innovation processes and discusses rationales for why these processes require support from the government. Link and Tassey (1988) offers a model to explore the impacts of the adoption of standards on the diffusion of advanced technologies, using numerically controlled machine tools as an historical example.[7] More recently, Tassey (2014) discusses the roles of standards in the knowledge economy. He argues that standards facilitate the production and use of technical information in knowledge-intensive industries, and so underpins the generation of economic benefits derived from technological change.

BEIS (2017) provides a summary of the evidence for the importance and value of measurement that was gathered whilst developing the UK Measurement Strategy. Following previous reviews by Tassey (1982) and Estibals (2012), the report reasons that measurement is an important infra-technology, essential for both process innovation and the development of new products. Measurement tools span across almost all sectors of the economy. For instance, measurement plays a crucial role in increasing productivity and reducing losses, as well as in underpinning regulation, trade[8], healthcare, navigation, communications, defence, and so on. Therefore, no single company or sector can internalise all the benefits from developing new or improved measurement science, which results in underinvestment and creates a market failure. Consequently, most governments invest in measurement science (metrology) to ensure that society has access to a national measurement infrastructure, along with direct support for the effective commercialisation of

[6] Greg Tassey, an American economist at the National Institute of Standards and Technology (NIST), has worked extensively in the economics of standards and high-tech industries. Peter Swann, an English economist, has worked extensively in the economics of innovation. Standards and metrology are among the topics that are commonly encountered in Swann’s research. And, lastly, Knut Blind, a German economist, has done extensive research on analysing the connections between standards, regulation, and innovation. His research has explored the impacts of regulation and standardisation on the innovative behaviour of companies, both at a micro- and macro-economic level. His research also focuses on the economics of intellectual property rights.

[7] Machine tools are a central element in the manufacturing of almost all physical products. They either produce the machines which in turn produce the final product, or they produce the final product directly. Numerical control is a method whereby machine tools can be controlled by programmed instructions using numeric or symbolic codes.

[8] Deloitte (2009) shows that £622 billion of the UK’s total trade relied on measurement.

other technologies whose development is sometimes impeded by measurement-related challenges.

The technology spillover argument presented above is the traditional rationale for public intervention to support measurement science, however, it is not the only argument in support of developing a national measurement infrastructure. The absence of such an infrastructure leads to coordination failures (Swann 2009), technological uncertainty (Swann 2009), and costly duplication arising from the need for a vast number of users to establish their own measurement standards (BIS 2015).

Swann (1999) contains an extensive literature review that establishes the mechanisms through which measurement activities and standards generate economic benefits. Based on Swann's framework, the main firm-level benefit mechanisms are:

- Supporting innovation and investments in new technologies: Swann (1999) reasons that innovation often takes a combinatorial form, that is, a new product offers a novel combination of product characteristics. Frenz & Lambert (2012) find that an industry's use of measurement is strongly correlated with its spending on R&D as a fraction of its turnover. Likewise, King et al. (2006) found a strong correlation between an industry's use of NMS services and the proportion of revenue generated via new and novel products.
- Increasing productivity: Measurement can affect productivity through three main channels - early detection of errors (Jula 2002; Kunzmann 2005; Allgair 2009), standardisation and division-of-labour (Smith 1776, Temple et al. 2005; Swann 2009), and greater exposure to international competition (Swann et al. 1996; Blind 2001; Blind and Jungmittag 2006; Aghion et al. 2009; Swann 2010).
- Enabling vertical differentiation in product markets: Information asymmetry is involved between buyers and sellers of goods (Akerlof 1970), and standards can play a significant role in reducing this asymmetry. The system of accreditation and certification promotes trust in the seller and reduces risk to the buyer, thereby, reducing transaction costs. That is, markets function more effectively when buyers are confident in the accuracy and reliability of information provided by suppliers.

King & Nayak (2023) develop a microeconomic model for the value created when measurement is used for conformity testing. The model shows how measurement information creates value by reducing mistakes in conformance testing (i.e., fewer false-positives and false-negatives). Additionally, introducing accurate calibrations as a perturbation on top of these measurement activities yielded a model for the value created from high-quality calibrations. Blind (2024) also investigates the role of the national quality infrastructure (regulation, standardization, metrology, conformity assessment and accreditation) in the general context of social and technological transformations. Most of this existing literature employs a microeconomic approach towards estimating the value of measurements. Hence, our study attempts to bridge this gap by taking a macroeconomic approach.

Our own study, as detailed in this report, is focused on the conformance testing aspect of measurement infrastructure. But we acknowledge that there are many other aspects of standards that are not covered by our study. These other aspects include: the role of standards international trade; the interplay between standards and intellectual property; the balance between standards and regulations; and implications for competition policy that have been explored previously in the economics literature. Hence, this literature review concludes with a list of aspects that are outside the scope of our study.

- Standards have an increasingly important role to play in a globalised economy. Promoting the widespread adoption of standards has the potential to improve the trade balance of a country and make its own markets more open, as Swann et al. (1996) has demonstrated for the UK. Ticona and Frota (2008) study the uptake of international standards and measurement techniques in Brazil, finding that 11% of growth in output from the studied industries (including steel and automotive tyres) was associated with certification.

- Blind & Jungmittag (2008) explore the impacts of patents and standards on macroeconomic growth. The study applies a growth model to pooled data from four European countries and found that the stock of patents and the stock of technical standards contributes to economic growth.
- Blind & Münch (2024) investigate how regulations, along with national and international standards, impact on both innovation inputs (e.g., R&D expenditure) and innovation outputs (e.g., patents). They find that on the one hand, international standards are positively associated with R&D expenditure and patenting, and that they outperform de-regulation and national standardisation. However, national standards are negatively related to patents and seem to localize economies geographically and slow down their evolution.

4 ANALYTICAL FRAMEWORK

This study takes a theory-based approach that incorporates the national quality infrastructure into a canonical macroeconomic model, known as the Solow Model. In this model, economic growth is driven by increasing the amount of capital per worker – a process known as "capital deepening". The model is intended to represent economic activity within the non-financial businesses, which account for about 77% of employment. Specifically, this study adapts the Solow model by making Total Factor Productivity (TFP) a function of the effort committed by specialist engineers to conformance testing (CT).

4.1 SOLOW MODEL

Early work by Solow in the 1950s showed how the per capita output of an economy grows, through "capital deepening", up to a steady state value of the capital intensity, after which capital accumulation just keeps pace with growth in the workforce.[9]

The Solow model combines an exogenously fixed savings rate with a simple model of production and capital accumulation. The model also assumes full employment, so that the workforce grows with the population. At the heart of the Solow model is an equation for change in the capital stock that models the investment funded by savings, as well as the depreciation that whittles away at the existing stock of capital.

Solow showed that capital accumulates until the economy's capital intensity reaches a steady state in which investment exactly offsets the effects of depreciation. The long run equilibrium is one in which GDP per capita plateaus out, so that the economy grows at the same rate as the population.

4.2 MALFUNCTIONS IN PRODUCTION

This study perturbs the well-established outcome of the Solow model by modifying its set-up to allow for the possibility of defective output due to malfunctions within the production process. Unless these malfunctions are addressed, the output of the economy will begin to decline. Hence, engineers are employed to search through the capital stock looking for malfunctioning machines that can then be reset.

To stop the reliability of the production process declining over time, some output goes towards paying engineers to undertake conformance testing (CT). Output from machines that fail the conformance tests is scrapped, and only output from machines that pass these tests

[9] Solow, R. M. (1956). A contribution to the theory of economic growth. *The Quarterly Journal of Economics*, 70(1), 65–94. doi:10.2307/1884513.

is allowed to enter the supply chain. One can interpret conformance testing as a way to counteract malfunctions that would otherwise be a source of “decay” within the capital stock. The central idea being that if a machine malfunctions, and isn’t reset, then this machine stops contributing to production. And, in such a situation, it’s as if the machine “died” before its time.

Readings from conformance tests come with an associated measurement error, and so the conformance testing process is prone to the possibility of type-1 errors (false-positives) and type-2 errors (false-negatives). In other words, sometimes the engineers needlessly reset a perfectly good machine (a type-1 error); at other times a conformance test incorrectly tells them that a malfunctioning machine is working fine and so they mistakenly let it pass (a type-2 error). The likelihood of a given machine not being reset can be characterised using a Poisson model, in which the expected number of resets in a year depends on the number of engineers who are employed to supervise production.

4.3 THE NATIONAL QUALITY INFRASTRUCTURE

The expansion of the Solow model, as detailed in this study, is based on introducing a series of new parameters that characterise additional aspects of the production process:

- The “transition rate” is the conditional probability that a previously well-functioning machine starts to malfunction.
- An engineer’s “span of control” is the amount of capital equipment (the value of the machines) that one engineer can be expected to supervise.
- The likelihood of a type-1 error is the conditional probability of a false-positive, which is associated with the “regret rate”.
- The likelihood of a type-2 error is the conditional probability of a false-negative, which is associated with the fraction of outputs being returned by unhappy customers, known as the “rebate rate”.

The following discussion outlines how each of these parameters can be pegged to a specific element of the national quality infrastructure.

Firstly, the Production Possibilities Frontier (PPF) is set by the best available production technique. Firms who are not using the best available technique (BAT) are likely to have a higher error rate than those that do. Moreover, the adoption of best practice will be influenced by whether firms are using standards as a matter of routine. Hence, the transition rate will be lower in situations where a high proportion of businesses adopt standards, thereby, ensuring that they use the best available techniques.

Secondly, standardisation plays an essential role in codifying routines on which the division-of-labour is based, and so helps to breakdown a complex production process into a series of simpler sub-processes. Let us also assume that the capital stock is composed of many different types of machinery, and that each type is associated with a particular sub-process. Standardisation allows production to be organised such that each supervisor oversees one part of the process. This promotes division-of-labour, which, in turn, enables specialisation. Hence, the greater the availability of standards, the greater the scope for specialisation, which then goes on to yield greater economies-of-scale.

- Skilled workers can turn their hand to many different parts of the process, but they can only become experts in one sub-process. Without specialisation an engineer must oversee many different parts of the production process, which limits their ability to develop a specialism. Specialisation combined with “learning-by-doing” means that skilled workers can perfect their understanding of one sub-process.
- The pace at which engineers can do their work will depend on the degree to which production processes can be broken into parts, proxied by the size of the stock of

standards that codify best practice. Realising economies-of-scale through specialisation allows an engineer to supervise many machines of the same type, thus increasing their span-of-control.

By these means, standardisation can increase the productivity of workers engaged in production activities. In the same way, standardisation will raise the productivity of those employed in measurement and testing roles. (Arguably, even more so given the very strong connection between standards and testing.[10])

Lastly, the conformance testing activities necessarily imposes a cost on the economy in return for almost guaranteeing that goods entering the supply chain conform to specification. Part of this cost is associated with paying the engineers to conduct the conformance tests. Another part of the cost comes from scrapping viable goods when the tests produce a false-positive. The quality and relevance of the measurement infrastructure sets the statistical power of these tests, and thereby determines the cost imposed by the conformance testing process. A good quality measurement infrastructure enables measurement errors to be kept to a minimum and reduces the impact of type-1 and type-2 errors.

4.4 EXTENDED SOLOW MODEL

In an expanded version of the Solow model, an engineer's span-of-control plays a similar role to other parameters in the classic Solow model, such as, growth rate of the workforce (i.e., the birth rate). This span-of-control is a fundamentally new parameter that sets the amount of capital that an engineer is capable of supervising.[11]

Our adapted version of the Solow model leads to a system of two equations: one for the evolution of the capital intensity and another for how the reliability of the production process evolves over time. The simplicity of the set-up means that the dynamics of the system can be pictured in a two-dimensional phase diagram, where the axes are the economy's capital intensity and the reliability of the production process.

To make quantitative inferences from the model, we will need to make estimates of the model's parameters. Some of the model's parameters are fundamental macroeconomic parameters, and much effort has already gone into determining the value of these parameters by the Office of National Statistics (ONS). Also, a useful theoretical relationship between these basic parameters is set out in the work of Thomas Piketty.[12] (In his book, Piketty also provides a wealth of historical data to support his thesis.)

It is possible to make reasonable estimates of things like the proportion of GDP spent on conformance testing. However, it's much more challenging to establish values for parameters pertaining to things like the likelihood of malfunctions in the production process or the pace of conformance tests. Thus, the parameters that feature in this study will be split into three classes:

1. Well established macroeconomic parameters that feature in the classic Solow model.
2. Parameters that are observable and for which we can make reasonable estimates.
3. Less observable parameters that have theoretical meaning but are difficult to directly measure, although, common sense may suggest a plausible range.

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[10] The UK Innovation Survey (UKIS) found that businesses use standards as a source of information at a higher rate compared to many other sources of information. On average, around 13% of businesses reported that standards were a highly important source of information. Moreover, the 'testing and analysis' industry is one of the industries for which the UKIS provides summary statistics; and around 34% of such businesses said standards were a highly important source of information. This percentage is markedly higher than for any other industry, and more than double the average percentage. For more information see the UK Innovation Survey (Statistical Annex): https://www.gov.uk/government/statistics/uk-innovation-survey-2019-main-report.

[11] The span-of-control (or "portion size") governs the employment of CT engineers within the economy in the same way as the teacher-to-pupil ratio governs the number of teachers employed by schools.

[12] There remains some academic debate but there a widely accepted values for such parameters.

So, how might we go about finding values for the unobservable parameters? The solution is to assume that the actual economy - the one that we currently inhabit - has (more or less) settled into its equilibrium and then use the established equilibrium relationships to find the value of the unobservable parameters.

The unobserved parameters are the unknowns in a series of equilibrium equations, which can be solved to find the value of the unobserved parameters using what we know about the values of the observed parameters. By this means, the model can be operationalised by finding a set of values for the unobservable parameters that yields an equilibrium outcome consistent with what's observed in the economy. That is, a numerical analysis can be used to search for values of the unknown parameters that yield the correct values for the known parameters.

Lastly, it should be understood that the status of the new parameters in our model is different from that of the basic parameters found in the classic Solow model. That is, the values ascribed to the new parameters are reasonable estimates, and so should be seen as sensible "ballpark" numbers. It will require a series of industry surveys to more accurately determine their values, and so this will become the topic of yet future empirical work.

5 THE BACKSTORY TO THE MODEL

It aids the overall exposition to begin by sketching a cartoon-like picture of an economy whose features fit the model used in this study. In particular, the following sketch helps to bring to life assumptions found in later sections. The features of this economy were chosen to simplify the mathematics and to make the model easier to explain - it isn't meant as a fair and accurate description of the UK economy. Rather, it's meant as the mathematical equivalent of a pruned-back experimental setup in which one parameter can be varied - whilst holding all else constant - to see what effect it has on the performance of the system.

5.1 A REPRESENTATIVE CITIZEN

The economy is made up of citizens who are both "workers" and "capitalists" because citizens are the owners of all the capital in the economy. Hence, their income is made up of wages from selling their labour and rents from leasing out capital. (Citizens' savings earn interest when banks lend out their deposits so that businesses can invest in new capital equipment). It's assumed that these citizens are much the same in terms of their capabilities and that the economy's wealth is evenly distributed amongst them. Hence, the analysis can be focussed on the welfare of a "representative citizen" who is replicated throughout the population.

5.2 THE FINANCIAL SECTOR

A fixed proportion of GDP (aggregate national income) goes into citizens' savings, which then passes through the financial sector and into the investments made by businesses. The financial sector plays an important role in facilitating transfers of money and wealth, but it isn't productive in the sense of producing outputs that contribute to the country's GDP. Hence, the financial sector doesn't feature in this study other than as a conduit for channelling savings into investments.[13] Nonetheless, the financial sector operates quietly in the background to ensure that aggregate savings always equal aggregate investment so that the circular flow of money is in equilibrium.

[13] The value created by the financial sector depends mostly on inputs of human capital and ready access to market information. Unlike non-financial businesses, increasing the capital (e.g., ITC equipment) used by banks and insurance companies won't significantly increase the value they create. Hence, the financial sector doesn't fit our story of "capital deepening", and so doesn't feature in this study.

5.3 TECHNOLOGICAL CHANGE

Labour productivity - along with the wage rate - is determined by the economy's Total Factor Productivity (TFP) and its capital intensity (that is, capital per worker). Economists believe that TFP growth is the result of additions to the stock of knowledge from purposeful investments in research and development (R&D). Historically, these investments have had a major effect on economic growth and supported a rise in peoples' living standards.

However, our study deliberately neglects the effect of the NMS programme on innovation within businesses, preferring instead to leave it to other dedicated studies to explore this route to impact. There is good evidence from such studies that the NMS labs contribute to growth and innovation among a group of 430 regularly supported firms, but this is outside the scope of this report.

Lastly, the UK's yearly TFP growth has been small compared to what it was in the years before the Great Recession of 2009. This suggests that a range of factors may be holding back innovation, so that the rate of technological change is somewhat slower than before 2009. Hence, we can imagine a typical worker being tasked with getting the most out of their technological inheritance, without a strong expectation of developing it very much further.

5.4 CAPITAL DEEPENING

This study focusses on the production of goods and services using a production process whose productivity depends on "capital deepening": Raising the amount of output per worker (within non-financial businesses) requires capital accumulation so that there's ever more capital per worker. Sometimes we will simply refer to "goods" but the outputs from production should be understood to involve both goods and services.

Most of the output from the non-financial businesses is ultimately consumed by those who work in the real economy. However, a part of peoples' income is saved (e.g., goes into pensions), and these savings are then used by companies to fund investments in productive capital (e.g., plant and machinery). In addition, some savings take the form of retained income that is reinvested in a business rather than given out to shareholders as dividends. Moreover, a baseline level of investment in new capital will always be necessary to offset the depreciation that whittles away at the capital stock. Once the capital intensity (capital per worker) has attained its steady state value, the per capita output also plateaus out so that the economy's GDP continues to grow at the same rate as its workforce.

5.5 MALFUNCTIONS IN THE PRODUCTION PROCESS

Suppose that immediately after its installation, a new capital item always works perfectly, producing output that fully conforms to specification. Unfortunately, as time goes on, it can start to malfunction, meaning that it slips into producing defective outputs. Moreover, such malfunctions aren't obvious to the workers engaged in production, and consequently the outputs it produces are only found to be defective after they have reached end-users. A buyer can secure a refund from the retailer who will then ask for compensation from the wholesaler. Such a wholesaler should be able to trace the defective outputs all the way back to the original producer. Malfunctions, of this kind, lower the economy's Total Factor Productivity (TFP), and ultimately decrease per capita output.[14] This suggests that it's in the self-interest of producers that a small part of their resources should go towards finding the malfunctioning machines and then resetting them, so that they no longer produce defective output.

5.6 CONFORMANCE TESTING CONDUCTED BY ENGINEERS

The malfunctions that occur within the production process can be detected, and corrected, by engineers who are employed to oversee production. Moreover, if there's a shortage of

[14] The term Total Factor Productivity (TFP) refers to the effectiveness with which labour and capital (known as the factors of production) can be combined to produce output.

homegrown engineers, the real economy can import technical services from abroad. Furthermore, the real economy can deal with any shortage of human capital by recruiting engineers from overseas; and access to this deep reservoir of global talent means that there's an almost unlimited supply of engineers.[15]

The tests conducted are "rigorous" (or demanding) in the sense that they almost never let through defective outputs. The downside of this level of rigour is that occasionally perfectly viable outputs are mistakenly scrapped. To some extent this is regrettable, but this is a sacrifice that society is willing to make for having high confidence in the output that enters the market.

5.7 BASIC CALIBRATION VERSUS PRECISE CALIBRATION

Measurements come with an associated measurement error. Suppose that this error is drawn from a distribution that's centred on zero and has a known standard deviation. Hence, there is a degree of uncertainty around the test results, by which is meant a degree of uncertainty in the measured values. Furthermore, for a given standard deviation (which sets the level of measurement uncertainty), there's a trade-off between the two types of mistakes: If the cost of type-1 errors decreases, then the cost type-2 errors will increase (and vice versa).

So, alongside the cost of employing engineers, there is also a cost due to mistakes in the conformance testing process itself. There are type-1 errors (false positives) in which perfectly good output is scrapped, as well as type-2 errors (false negatives) in which malfunctions go undetected, so that machines continue to produce their defective outputs. The effectiveness of conformance testing, and an engineer's span-of-control depends on the infra-technology that underpins the economy's national quality infrastructure. This infra-technology is a public good and its quality depends on R&D and maintenance work performed by publicly funded institutions, such as, the NPL.

5.8 A RELIABILITY TAX TO PAY FOR CT ENGINEERS

Conformance testing is not directly productive, rather it provides confidence to both buyers and sellers that what has been produced meets any regulations and is of dependable quality. So, in an important sense, conformance testing takes place outside the real economy, much like employing a police force and judiciary to ensure "good order". The people employed in such jobs create something useful to society (confidence), but it cannot be bought and sold in the same way as products can. Hence, conformance testing diverts resources that – in a perfect world – would be used for production. In other words, conformance testing benefits society by ensuring that production processes are reliable, but it comes at a cost, because it uses resources that would otherwise by deployed elsewhere.

The reason for this cost is that CT engineers won't work for free, and so they require a small fraction of the revenue from production as payment. Moreover, from the perspective of setting up the model, there needs to be a direct cost from employing CT engineers, and so the wages paid for their services play this role in the model.

Finally, citizens must pay for the benefit of having reliable production processes and the security of knowing that the goods they buy probably aren't defective. There are many institutional arrangements through which the payments could be collected but the simplest to model mathematically amounts to a flat tax on citizens' incomes. Indeed, at an aggregate level, the complexity of the different institutional arrangements washout to result in the same claim on citizens' incomes. So, whilst this isn't a literal tax on peoples' incomes, it resembles an import duty paid by domestic producers and consumers in the form of somewhat higher costs and higher prices. Moreover, this "reliability tax" is freely paid by producers, because it is in their enlightened self-interest to avoid supplying defective outputs to their customers. In

[15] Engineers operate in a global labour market.

other words, the spending on conformance testing resembles a tax from a macroeconomic perspective but it is paid voluntarily for self-interested reasons. (It is rather like a “tribute” or “sacrifice” that is paid voluntarily by producers to keep the harmful effects of decay or entropy under control.)

6 STATISTICS FOR THE REAL ECONOMY

This study develops a theoretical model that builds out from conventional macroeconomics. In terms of fitting this model to data, the focus is the UK’s “real” economy during the five-year period from 2015 to 2019. The reasons for this focus are as follows:

Firstly, the model developed in this study fits the activities of private organisations excluding those working in finance. On aggregate, the activities of these organisations correspond to the “real” economy: the UK’s non-financial businesses, as well as private not-for-profit organisations (e.g., charities). The “real” economy accounts for 77% of total employment with most of the remaining 33% employed by government organisations (e.g., the civil service).

This concept of the “real” economy maps neatly onto data that is routinely collected by the ONS (Office of National Statistics). Firstly, there’s the Annual Business Survey (ABS), which is a yearly ONS survey of the UK’s non-financial businesses. Secondly, the ONS provides a yearly bulletin for the rate-of-return on the capital employed by non-financial businesses. Lastly, the ONS has the Business Register and Employment Survey (BRES), providing data on the number of jobs broken down by sector. Together these datasets provide much of what’s needed to estimate the parameters in the model.

Let us now explain the choice of time period. Firstly, the period 2015 to 2019 ends before the economic shocks associated the Covid pandemic of 2020-21. (The issue being that the long run equilibrium characterised in our study probably doesn’t hold too well during such a crisis.) Secondly, from an earlier study, we have readymade estimates of measurement activity, which gives us estimates of some key parameters for 2017. Therefore, for a mixture of theoretical and pragmatic reasons, we focus on the period from 2015 to 2019, with 2017 being the middle year of this period.

6.1 GROSS CAPITAL STOCK

The level of production in the real economy depends on the stock of productive capital. We can think of this as the machinery and equipment used by businesses in their production activities. This excludes wealth tied up in land and property.

There is a technical distinction between “gross capital” and “net capital”.

- The gross estimate is based on the price of the assets when purchased.
- The net estimate takes into consideration the effects of depreciation.

When considering a country’s productive capacity in a given year, it isn’t relevant that a portion of its assets are halfway through their technical lifetimes. Providing that a capital item works properly, it can be rented out at much the same rate as a brand-new equivalent. Therefore, when considering how much a country can produce each year, it is the “gross capital” that counts.

The economy’s “capital ratio” is the value of the capital stock as a percentage of its Gross Domestic Product. As we aren’t considering the whole economy, the Gross Value Added (GVA) should be substituted for Gross Domestic Product (GDP).

In 2017, the gross capital employed by non-financial businesses was worth £3,362 billion. Since the GVA generated by their activities was £1,222 billion (at basic prices), this capital stock was worth 275% of this yearly GVA. Moreover, for the five-year period from 2015 to 2019, the capital ratio also averaged 275%.

6.2 RATE OF RETURN AND DEPRECIATION RATE

The level of investment depends on the interest rate, and the interest rate is determined by the gross rate-of-return (rental rate), along with the rate of depreciation for capital equipment.

The capital market will settle into an equilibrium where the owner of a capital item is indifferent between (1) earning income by renting it out and (2) selling their asset and then putting the proceeds in the bank to earn interest. It follows that the rental rate must be such that the rate-of-return (the rental rate minus the depreciation rate) equals the interest rate. The bank pays interest to savers by lending out their money to businesses to fund investments. Lastly, the long-run growth rate for the stock market (FTSE) is around 7% per annum, and so with a long-run inflation rate of 2% this suggests a real interest rate of around 5% per annum.

In 2017, the gross rate-of-return and the depreciation rate were as follows:

- The annual gross rate-of-return for non-financial businesses was 11.0%, which comes from dividing the gross operating surplus (£369.5 billion) by the gross capital employed (£3,362.4 billion).
- The average annual depreciation rate for non-financial businesses was 4.8%, which comes from dividing the capital consumed (£161.3 billion) by the gross capital employed (£3,362.4 billion).

For the five-year period 2015 to 2019, the gross rate-of-return averaged 11.2%, and the depreciation rate averaged 4.8%. Subtracting the depreciation rate (4.8%) from the gross rate-of-return (11.2%) yields an interest rate of 6.4%.

Note that this interest rate of 6.4% is higher than the discount rate of 3.5% found in HMT's Green Book. The reason for this is that the discount rate is based solely on the rate of time preference[16], whereas there are risks associated with lending to businesses. Consequently, the interest rate of 6.4% is formed by adding a risk-premium to the base-rate. This corresponds to the rate at which a portfolio of similarly risky investments can earn interest.

6.3 SIZE OF THE WORKFORCE

In 2017 the UK had a population of 66.1 million and a workforce of 32.1 million (based on total employment). Hence, about half the population is in the workforce, with the youngest quartile in education, and the oldest quartile in retirement.

According to data for 2017 from the Business Register and Employment Survey (BRES), total employment by the UK's private sector (excluding finance and insurance) was about 25.1 million, which is around 78% of total employment in the UK.[17] According to data for 2017 from the Annual Business Survey (ABS), employment costs for non-financial businesses were £642.5 billion, implying an average wage of £25.6 thousand. Note that including mixed income (allowing for payment in the form of shares) would tend to raise the income that goes to labour.

Data from the BERS was used to estimate an average growth rate for employment amongst the non-financial businesses: In 2019 the number of people in employment was 25.5 million, whereas in 2015 it had been 24.0 million. Hence, the annualised growth rate for the five-year period from 2015 to 2019 is about 1.5% and, in our model, this sets the economy's growth rate.[18]

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[16] Citizens prefer current consumption to future consumption. The discount rate of 3.5% is an estimate of what is needed to compensate citizens for differing some of their consumption by a year.

[17] The distinction between 'employment' and 'employees' is that employment includes the self-employed.

[18] The calculation is $[25.5/24.0]^{1/4} - 1 = 1.5\%$.

6.4 GROSS VALUE ADDED

Gross Value Added (GVA) is the sum of the payments to workers, the rents paid to the owners of capital (gross profits), and the indirect taxes on production. Hence, GVA is an income-based measure of the output from businesses. The central idea is that the income earned through production activities equates to the value of the goods and services produced. (Often we will just refer simply to "goods", but the real economy also provides services to businesses engaged in production, transportation, and distribution.)

GVA tends to be valued at "basic prices", meaning that it excludes indirect taxes on products (e.g., VAT), but it includes indirect taxes on production. Removing the indirect taxes on production (which are only a few percent) gives us an estimate of GVA at "factor cost", meaning that it exactly equals the sum of the payments made to labour and capital.

It is assumed that production is perfectly competitive, and so there are no supernormal profits. The "gross operating surplus" corresponds to money paid to the owners of capital as rents. So, this is the accountant's concept of profit rather an economist's concept of profit.

The payments made to labour include not only wages but also the "mixed income" paid to employees who own shares in their businesses. That is, part of their remuneration is in form of shares from which they can receive dividends or sell on to other investors. It follows that if we were to only focus on wages, then we would underestimate labour's share of income. In practice, "mixed income" is what remains once gross operating surplus and indirect taxes have been subtracted from GVA.

According to data from the Annual Business Survey (ABS) for 2017, the GVA of non-financial businesses (at basic prices) was £1,222.4 billion. Using information from other datasets the GVA from non-financial businesses can be split into its four components as follows:

- Wages amounted to £642.5 billion but this doesn't include the "mixed income" paid to employees with a stake in their businesses. (From the BRES.)
- The gross operating surplus (profits) amounted to £369.5 billion. (From ONS's yearly bulletin on the profitability of UK companies.)
- Indirect taxes account for 1.6% of GVA, which amounts to £19.6 billion. (From ONS's Supply-Use tables for the whole economy.)
- The "mixed income" amounted to £190.8 billion, and so accounts for about 23% of the payments received by labour. This is the residual between GVA at basic prices and the sum of the three components given above: £1,222.4 bn - £642.5 bn - £369.5 bn - £19.6 bn = £190.8 bn.

Putting all this together implies that GVA at factor cost was around £1,203.8 billion. It also has the following implications:

- In 2017, capital's share of income was 30.7%.[19] For the five-year period 2015 to 2019, capital's share of income averaged 31.1%, and so labour's share of income averaged 68.9%.
- Since labour's share of income was 68.9% and GVA was £1,203.8 billion, this implies that wages and mixed income amounted to £829.4 billion. Since there are 25.1 million employees in the real economy in 2017, it follows that an employee earns about £33 thousand per year from working.

6.5 THE SAVINGS RATE

The savings rate is the percentage of total output (total income) that isn't consumed but, rather, is set aside to fund gross investment (gross fixed capital formation). Such investment

[19] The calculation is £369.5 billion ÷ [£1,222.4 billion × (1 – 1.6%)] = 30.7%.

is used to buy machinery and equipment and thus to build up the capital stock. Statisticians tend to refer to gross investment as Gross Fixed Capital Formation (GFCF).

For the real economy, savings are equivalent to income from wages and rents that is then reinvested in businesses. The equilibrium condition for the circular flow of money requires that gross savings equal gross investment, and so gross savings can be inferred from gross investment.

Total output corresponds to GVA at factor cost, and for the real economy this was £1,203.8 billion in 2017. So, to find the savings rate, we next need to estimate gross investment by non-financial businesses.

Gross investment equates to the net change in the capital stock plus the capital consumed through depreciation. An accurate measure of the change in the capital stock can be found by applying capital deflators to ensure values are comparable across years.

Gross investment can be estimated using data from ONS's annual statistical bulletin on the profitability of UK companies. The calculations for 2017 are shown in Table 1.

Table 1: Estimating Gross Investment in 2017 for the Real Economy

Calander Year	Gross Capital Employed	GFCF Deflator	Gross Capital Employed	Change in Capital Stock	Capital Consumed through Depreciation	Gross Investment (GFCF)
	£ billions Nominal	(2017 = 100)	£ billions Real	£ billions Real	£ billions Real	£ billions Real
2016	3168.4	97.1	3263.0	.	.	.
2017	3362.4	100.0	3362.4	99.4	161.3	260.7

So, in 2017, gross investment was £260.7 billion and GVA at factor cost was £1,203.8 billion. Hence, the savings rate was 21.7%. The same set of calculation can be repeated for each year of the period under consideration. For the five-year period 2015 to 2019, the savings rate averaged 20%.

There is a well-established economic theory through which the savings rate can be rationalised. Namely, the approach introduced by Frank Ramsey in the 1920s combining intertemporal substitution, a dynamic utility maximisation problem, and mathematical techniques from optimal control theory.[20] However, to keep the analysis as simple as possible, in our study the savings rate will be taken as fixed parameter of the economy that's determined by the basic psychology of the population (i.e., peoples' capacity for deferred gratification). There's potential for the framework developed in our study to be extended in ways that would endogenise the savings rate, but that isn't attempted here, and so this is left as a topic for further work.

6.6 THE BASIC PARAMETERS AND BASIC VARIABLES

For the five-year period 2015 to 2019, the basic parameters for the real economy were as follows:

[20] Ramsey, F. P. (1928). "A Mathematical Theory of Saving". The Economic Journal. 38 (152): 543–559. doi:10.2307/2224098.

- The interest rate was 6.4%.
- The depreciation rate for capital was 4.8%.
- The growth rate of the workforce was 1.5%.
- Capital's share of income was 31.1% and Labour's share of income was 68.9%.
- The savings rate was 20%.

During this period, the basic variables were as follows:

- The GVA at factor cost was £1,203.8 billion. (The output of the real economy.)
- Employment in the real economy was 25.1 million.
- The gross capital employed by non-financial businesses was worth £3,362 billion.
- The wage rate was £33 thousand, and the gross rate of return on capital was 11.2%.

7 AGGREGATE PRODUCTION AND FACTOR COSTS

This section sets up the traditional model of production based on the presumption of perfect competition along with perfect production processes that somehow never malfunction. Later section of this report will build out from this model to allow for the possibility of imperfect production processes that occasionally produce defective outputs.

7.1 THE PRODUCTION FUNCTION

This subsection introduces a production function for the real economy in which production is based on combining the economy's factors of production (labour and capital) using a fixed level of technology.

To set up the model, it's helpful to define the following sets: Let $(0,1) \equiv \{x \in \mathbb{R} : 0 < x < 1\}$ and $\mathbb{R}_{++} \equiv \{x \in \mathbb{R} : x > 0\}$, where $\mathbb{R}$ denotes the set of real numbers.

Let the economy's aggregate output (Y) be described by a Cobb-Douglas production function in which the factors of production are labour and capital: The aggregate production function is as follows:

$$Y = F(L, K) := AL^{\alpha}K^{\beta} , \tag{7-1}$$

where $L \in \mathbb{R}_{++}$ and $K \in \mathbb{R}_{++}$ are variables representing labour and capital, respectively. The constant parameters of this function are α, β and A. The elasticities of labour and capital are $\alpha \in (0,1)$ and $\beta \in (0,1)$, respectively. Furthermore, the economy's Total Factor Productivity (TFP) is $A \in \mathbb{R}_{++}$, and represents the power of the technology through which labour and capital are brought together to generate output.

- '*Y*' represents the aggregate revenue generated by selling the economy's output. That is, it's the Gross Value Added (GVA) generated by the UK's non-financial businesses. Lastly, everything is in terms of constant prices.[21]
- '*L*' is taken to represent a large pool of unskilled workers. Further suppose that these workers are undifferentiated, and so interchangeable from the perspective of employers.
- '*K*' represents the stock of productive capital (such as, production plants, ICT equipment, and machinery), which is recorded as fixed assets in company balance

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[21] A low rate of constant inflation could be added into the model without any theoretical difficulties and would not change the results.

sheets. It does not include the value of land and property owned by the population as this isn't productive in the way that machinery and equipment is productive.

Suppose that the economy is operating near the Production Possibilities Frontier (PPF), meaning that any easily achievable efficiency gains have already been exploited. Most specifically, all production processes are operating at the efficient-scale, so that there are no more economies-of-scale to be realised. Consequently, if the economy expands, it does so by replicating production plants that already exist, rather than by expanding the existing ones. This means that if labour and capital were both to double, then so would the economy's output, and this condition is referred to as "constant returns-to-scale". Mathematically, this means that doubling L, K results in a doubling of $F(L, K)$:

$$F(2L, 2K) = 2F(L, K).$$

Which is equivalent to assuming that the function $F(L, K)$ is homogenous of degree one. Moreover, as the model is based on a Cobb-Douglas production function, it follows that:

$$F(2L, 2K) = 2^{\alpha+\beta}F(L, K).$$

And, thus, constant returns-to-scale requires that: $\alpha + \beta = 1$.

The argument above shows that assuming constant returns-to-scale is equivalent to assuming that the production function is homogeneous of degree one, and so Euler's theorem yields:

$$Y = F(L, K) = \left(\frac{\partial Y}{\partial L}\right) \times L + \left(\frac{\partial Y}{\partial K}\right) \times K \,;$$

where the marginal product of labour (MPL) is

$$\frac{\partial Y}{\partial L} = \frac{\alpha A K^{\beta}}{L^{1-\alpha}} = \frac{\alpha Y}{L} \,;$$

and the marginal product of capital (MPK) is

$$\frac{\partial Y}{\partial K} = \frac{\beta A L^{\alpha}}{K^{1-\beta}} = \frac{\beta Y}{K} \,.$$

Finally, the economy's labour productivity becomes:

$$\frac{Y}{L} = (\partial Y/\partial L) + (\partial Y/\partial K) \times \frac{K}{L} \,,$$

where Y/L is the output per worker, and K/L is the capital per worker. Let K/L be referred to as the economy's capital intensity. It will be seen that Y/L is also the yearly income of the "representative citizen", as well as the economy's labour productivity.

7.2 FACTOR MARKETS

As discussed below, labour and capital are paid their respective marginal products. This means that the wage rate corresponds to the value of the output produced by the last worker employed, and the rental rate is determined by the return on the last unit of capital added to the capital stock. Let "MPL" and "MPK" denote the marginal product or labour and the marginal product of capital, respectively. Let w denote the wage rate for labour, and r denote the rental rate for capital.

Suppose that there is full employment and a competitive labour market (e.g., no "closed shops" or restrictive practices). In this situation, the wage rate is determined by the value of the output attributable to the efforts of the last worker employed (the marginal worker). This means that workers receive, as wages, the value of the extra output generated by the last worker employed, referred to as the marginal product of labour (MPL). Furthermore, any employee can play the role of this 'marginal worker', by implicitly threatening to withdraw their labour, and so this implicit threat sets the equilibrium wage in the economy at large. Therefore, the wage rate in the labour market adjusts until it equals the economy's marginal product of labour:

$$w = \mathrm{MPL} = \partial Y/\partial L = \alpha Y/L. \tag{7-2}$$

Notice that the rate wage is proportional to the economy's labour productivity (Y/L). Also, because $0 < \alpha < 1$, the average product of labour (that is, Y/L) must be somewhat larger than the marginal product of labour (that is, $\partial Y/\partial L$).

The owners of capital receive as rents the value of the extra output generated by the last one million pounds of capital items rented out. This is the value of the extra output generated by the last capital item used in production. It follows that the rental rate becomes:

$$r = \mathrm{MPK} = \partial Y/\partial K = \beta Y/K\,. \tag{7-3}$$

The average product of capital (Y/K) must be somewhat larger than the MPK ($\partial Y/\partial K$) given that $0 < \beta < 1$. Notice that the economy's capital ratio (K/Y) is inversely proportional to the rental rate, r:

$$K/Y = \beta/r\,. \tag{7-4}$$

Finally, as already explained, the economy's labour productivity is given by the following formula:

$$\frac{Y}{L} = (\partial Y/\partial L) + (\partial Y/\partial K) \times \frac{K}{L}\,.$$

Substituting the wage rate and rental rate for MPL and MPK, respectively, gives:

$$\frac{Y}{L} = w + r \times \frac{K}{L}\,. \tag{7-5}$$

Where, Y/L is the income of a typical citizen, and K/L is the amount of capital "owned" through their pension pot. Notice that this is a reinterpretation of the equation of labour productivity as being an equation for the consolidated income of a typical citizen.

7.3 THE INCOME OF THE REPRESENTATIVE CITIZEN

The citizens receive part of their income in wages and part of their income in dividends - as they are also the ultimate owners of all the capital in the economy. Note that the citizens fund investments using the portion of income that isn't used for consumption. These savings are put into pension funds that financial institutions then transform into debt and equity to fund the investments made by businesses. Consequently, citizens are, in a sense, both workers and capitalists (although, for the most part, they are rather passive capitalists who don't exercise much influence over the firms).

It's convenient to suppose that wealth and earnings are evenly divided across the population, so that people enjoy a similar level of prosperity. Clearly, this is a simplification of a more complex reality. There are disparities in wealth and income across the population and across the generations. This study does not deny the reality of distributional issues or inequalities, but it's not the subject of this analysis.

Most specifically, let K/L represent the amount of capital owned by a typical citizen. In this situation, it's possible to think of a single representative citizen whose wealth tracks that of the general population. (More concretely, the "representative citizen" is someone of pensionable age but who nonetheless still works. Hence, their wealth tracks the median for the population.)

The total payment to labour is $wL = \alpha Y$ and the total payment to capital is $rK = \beta Y$. Dividing these totals through by total income, Y, gives the share of output going to each factor of production:

$$\alpha = (wL)/Y \; ; \tag{7-6}$$

$$\beta = (rK)/Y \; . \tag{7-7}$$

Hence, the parameter α corresponds to the proportion of total output going to labour as wages (and mixed income); and the parameter β corresponds to the proportion of total output that goes to the owners of capital in the form of rents. Typically, labour receives around two-thirds of the total income that is generated by production activities and capital gets the remaining third in the form of rents or dividends. Based on ONS data for the real economy, we found that $\alpha = 68.9\%$ and $\beta = 31.1\%$. (Own calculations based on ONS data.)

Lastly, notice that $wL + rK = \alpha Y + \beta Y$, where $\alpha + \beta = 1$. Hence, it follows that: $wL + rK = Y$. This implies the aggregate income received by citizens equals the aggregate output of the economy, with this income being split between wages and rental income according to the relative size of parameters α and β.

7.4 A COMPETITIVE PRODUCT MARKET

Suppose that the product market is perfectly competitive, meaning that government officials (e.g., Competition and Markets Authority) don't allow companies to accrue market power and thus the "invisible hand" ensures that the price of a good equals its marginal cost.

The aggregate profit from production is the aggregate revenue minus the aggregate cost:

$$\text{aggregate profit} = \text{aggregate revenue} - \text{aggregate cost}$$

It can be shown that, together, constant returns-to-scale and competitive factor markets imply the product market will be perfectly competitive in the sense that the price of what's produced equals the cost of production, and so there are no supernormal profits. The argument runs as follows: Firstly, the total cost of production can be written as follows:

$$\text{aggregate cost} = wL + rK$$

Secondly, combining Euler's theorem with constant returns-to-scale implies that the total value of the output can be written as follows:

$$\text{aggregate revenue} = (\partial Y/\partial L) \times L + (\partial Y/\partial K) \times K.$$

Finally, with competitive factor markets, wages and rents are $w = \partial Y/\partial L$ and $r = \partial Y/\partial K$. Therefore, aggregate revenue equals aggregate cost:

$$\text{aggregate revenue} = wL + rK = \text{aggregate cost}$$

Since profit is defined as revenue minus costs, this means that entrepreneurs don't make any profits. Nonetheless, entrepreneurs can still earn an income by contributing to their business as workers (e.g., managers). Hence, entrepreneurs are best thought of as a special kind of worker, whose productive activities are as much beholden to capital as their employees. As discussed in the next section, this setup does not preclude the existence of economic rents, but such rents go to the owners of capital.

7.5 THE INTENSIVE FORM OF THE PRODUCTION FUNCTION

As already discussed, the production function is $Y = AL^{\alpha}K^{\beta}$, where Y is GVA of the real economy, L is labour input, and K is the capital employed. Based ONS data for 2017, estimates for the values of these three variables are as follows:

- The GVA of the real economy (at factor cost) was £1.2 trillion.

- The UK's workforce numbered 32 million workers, of which 25 million were employed in non-financial organisations in the private sector (which is a proxy for the real economy).
- The UK's stock of produced assets (equipment and machinery) was valued at £4.5 trillion, of which £3.5 trillion can be ascribed to the real economy given that it accounts for 78% of the UK's employment.

Lower case letters will be used to denote the "intensive" form of a variable, by which we mean that per capita version: Let $y = Y/L$ and $k = K/L$ denote per capita output and capital intensity, respectively.

From the bullet points above, it follows that the output per worker (labour productivity) is £48 thousand; and capital per worker (capital intensity) is £140 thousand. Where, to be clear, both these values are for the real economy: $y = £48$ thousand; and $k = £140$ thousand.

Assuming constant returns-to-scale means that the production function can be written in per capita terms as:

$$y = f(k)\,; \qquad \text{7-8}$$

where the intensive form of the production function is given by:

$$f(k) := Ak^{\beta}\,. \qquad \text{7-9}$$

From this it follows that labour productivity, y, is an increasing function of capital intensity, k. Before launching into the analysis of wages and rents, it's helpful to review the properties of $f(k)$, as well as the those of its first and second derivatives, denoted $f_k(k)$ and $f_{kk}(k)$, respectively. It is convenient to introduce the following subscript-based notation for the derivatives of $f(k)$:

$$f_k(k) := \frac{df}{dk} = \beta A k^{-\alpha}\,, \qquad \text{7-10}$$

$$f_{kk}(k) := \frac{d^2 f}{dk^2} = -\alpha\beta A k^{-(1+\alpha)}\,. \qquad \text{7-11}$$

Where the subscript '*k*' is being used to denote the first derivative with respect to '*k*'; and '*kk*' is being used to denote the second derivative with respect to '*k*'.

As much of the analysis depends on the properties of $f(\cdot)$ and $f_k(\cdot)$, a list of their key characteristics is provided: Firstly, as the first derivative is $f_k(k) > 0$ and the second derivative is $f_{kk}(k) < 0$, it follows that $f(k)$ is an increasing concave function of k. Secondly, the elasticities of $f(k)$ and $f_k(k)$ with respect to k are such that:

$$f_k(k)/f(k) = \beta/k > 0\,,$$

$$f_{kk}(k)/f_k(k) = -\,\alpha/k < 0\,.$$

(The first of these identities turns out to be particularly important, and so will be revisited as the end of this subsection.) Finally, as $f_k(k)$ is a strictly decreasing function of k, it has an inverse: $f_k^{-1}[f_k(k)] = k$. The closed-form expression for this inverse is as follows:

$$f_k^{-1}(r) := \left(\frac{\beta A}{r}\right)^{1/\alpha} \qquad \text{7-12}$$

Using the equation for per capita output, $f(k)$, and the assumption that $\alpha + \beta = 1$, the marginal products of labour and capital can be expressed as functions of the economy's capital intensity. From this, it follows that wages and rents are also determined by the economy's capital intensity. Firstly, since $\partial Y/\partial L = \alpha y$, the wage rate, w, becomes:

$$w = \frac{\partial Y}{\partial L} = \alpha f(k)\,, \qquad \text{7-13}$$

where $f(k)$ is an increasing function of the economy's capital intensity, k. This shows that if the capital stock grows faster than the workforce, then wages rise, whereas, if growth in the workforce outpaces the rate of capital accumulation, then peoples' wages will begin to decline. Secondly, since $\partial Y/\partial K = \beta y/k$, the rental rate, r, becomes:

$$r = \frac{\partial Y}{\partial K} = f_k(k)\,, \qquad \text{7-14}$$

where $f_k(k)$ is a decreasing function of the economy's capital intensity, k. This is in keeping with the intuition that, as capital becomes less scarce, it's 'price' decreases.

If $k \to 0$, then $f_k(k) \to \infty$, which implies that the marginal product of capital (MPK) would become infinite if the capital stock were to almost vanish. Lastly, notice that if $k \to \infty$, then $f_k(k) \to 0$, which implies that the marginal product of capital (MPK) would be negligible if the capital stock were to become so vast that machines lost all scarcity.

Finally, a useful insight from the analysis above is that because the rental rate is $f_k(k)$ and the proportion of output that goes to capital is β, it must follow that:

$$f_k(k)k = \beta f(k)\,. \qquad \text{7-15}$$

Since $\beta = 1 - \alpha$, this expression can be rewritten as:

$$f(k) = \alpha f(k) + f_k(k)k\,. \qquad \text{7-16}$$

On one level, this is just a mathematical identity. However, it also has an important economic interpretation given that $\alpha f(k)$ is the wage rate and $f_k(k)$ is the rental rate: $r = f_k(k)$. It can be seen the LHS gives the income of the representative citizen, and the RHS shows the split between wages and rental income: The first term is their wage, w, and the second term is their rental income, rk. (Recall that capital is ultimately owned by population. Hence, k is the average amount of capital per citizen, as well as being the economy's capital intensity.)

Some rearrangement yields a formula for the economy's capital ratio:

$$\frac{k}{f(k)} = \frac{\beta}{f_k(k)}\,. \qquad \text{7-17}$$

This formula gives the value of the stock of capital as a multiple of the yearly output of the economy.

7.6 DEPRECIATION, RENTS AND THE INTEREST RATE

Each year a constant proportion of the capital stock expires due to depreciation. Let δ denote the annual depreciation rate applied to the capital stock. Furthermore, in this study, it is convenient to imagine that δ is determined by a fixed "breakage rate" rather than an item's technical lifetime.

The capital market adjusts until the marginal return on net investment (i.e., the marginal product of capital minus the depreciation rate) equals the real interest rate: $f_k(k) - \delta = \imath$, where $f_k(k)$ is the marginal product of capital, δ is the depreciation rate, and $\imath$ is an real interest rate that combines the risk-fee rate with a risk premium.[22] That is, the cost-of-capital (or equivalently the rental rate) is $\imath + \delta$. This means that the gross spending on new capital

[22] Theoretically, the long run average for the risk-free rate is the 3.5% rate of time preference given in HMT's Green Book; and it can also be thought of as the interest of Treasury bonds. However, the rate at which businesses can borrow to fund investments will necessarily be higher than this because lenders must be compensated for the risk that the business becomes insolvent.

items (gross investment) will rise until the amount of capital per worker, k, is such that the return equals the cost. Thus, we arrive at the following equilibrium condition for the marginal product of capital:

$$\frac{\partial Y}{\partial K} = f_k(k) = \imath + \delta\,. \qquad 7\text{-}18$$

This says that the marginal product of capital equals the rental rate for capital items, where this rental rate corresponds to the interest rate plus the depreciation rate. The implication is that those who own capital items, and rent them out, need to be paid a little more than the interest rate because the income they receive needs to cover the cost of depreciation.

Notice that in this model the interest rate is ultimately determined by the size of the capital stock relative to the size of the workforce: If the capital intensity is high, then machines are not too scarce, and so they tend to be rented out cheaply. On the other hand, if the capital intensity is low, then there is strong competition for machines, meaning that they can be rented out at a high rate. Because the rental rate and the interest rate must track one another in the capital markets, it follows that the capital intensity ultimately determines the interest rate through its effect on the rental rate. The implication is that if the level of investment is insufficient to counteract depreciation, then this leads to a decline in the size of the capital stock, which subsequently feeds through to higher interest rates in the future. Alternatively, if there has been strong investment so that the capital stock increases, then this is will lead to lower interest rates in the future.

As discussed, plausible values for the interest and depreciation rates are $\imath = 6.4\%$ and $\delta = 4.8\%$, respectively. Hence, we arrive at a rental rate of $r = \imath + \delta = 11.2\%$.

Using these results, we can find a formula for the gross capital stock as a percentage of Gross Value Added, Y. Firstly, it's already been shown that: $\beta = (rK)/Y$, which implies that the capital stock as a percentage of GVA must be given by the following formula:

$$\frac{K}{Y} = \frac{\beta}{r} \qquad 7\text{-}19$$

Secondly, it's already been established that: $r = \imath + \delta$. Therefore, the value of the capital ratio can be found using the following formula:

$$\frac{K}{Y} = \frac{\beta}{\imath + \delta} \qquad 7\text{-}20$$

Substituting $\beta = 31.1\%$ and $\imath + \delta = 11.2\%$ into the formula above, gives us $K/Y = 278\%$. This percentage is almost identical to the estimate of gross capital as a percentage of aggregate income (GVA) that came from data for the UK's non-financial businesses.

7.7 TOTAL FACTOR PRODUCTIVITY

The economy's TFP (A) can be expressed as a weighted geometric mean of the MPL (wage rate) and the MPK (rental rate), where the weights are the indices for labour (α) and capital (β). Specifically, it can be shown that:

$$A = \left(\frac{w}{\alpha}\right)^{\alpha}\left(\frac{r}{\beta}\right)^{\beta}. \qquad 7\text{-}21$$

The proof is as follows:

Proof. Firstly, from $y = Ak^{\beta}$ and $w = \alpha y$, we get: $w/\alpha = Ak^{\beta}$. Next, it's already been shown that $k = f_k^{-1}(r)$, where $f_k^{-1}(r) := (\beta A/r)^{1/\alpha}$. So, using this inverse function to substitute for k yields $w/\alpha = A(\beta A/r)^{\beta/\alpha}$, which can be rewritten as: $(w/\alpha)^{\alpha} = A(\beta/r)^{\beta}$. A little further rearrangement then completes the proof. ■

This formula for the TFP can be evaluated using estimates of the quantities involved:

$$\text{TFP} = \left(\frac{\text{£33 thousand}}{68.9\%}\right)^{68.9\%}\left(\frac{11.2\%}{31.1\%}\right)^{31.1\%} = 10.466$$

Note that this is really an estimate of the "effective TFP" rather than the maximum achievable TFP. That is, since this is the TFP as observed in the real economy, it must already account for defective output and mistakes in the testing process. The consequence of such errors is that wages and rents are a little lower than they would be if production technology always operated flawlessly. This issue will be discussed at length in subsequent sections of this report.

8 PIKETTY'S "LAWS" OF CAPITALISM

A close descendent of the Solow model is the model used by Thomas Piketty in his book *Capital in the 21st Century* (2014). Like Solow, Piketty assumes that the savings rate is an exogenously set parameter of the economy, and his claim that capital's share of income is fixed is in keeping with one of Kaldor's stylised macroeconomic facts.

Through his book, Piketty expounds what he calls the first and second "laws" of capitalism:

1. The first law defines capital's share of national income as aggregate gross profit divided by national income (GVA) and claims that capital's share of national income is a fixed parameter of the economy.
2. The second law says that when the circular flow of money is in equilibrium, gross savings must equal gross investment and claims that the savings rate (which is savings as a proportion of national income) is a fixed parameter of the economy.

Hence, the first law defines the "capital coefficient" (or "capital ratio"), and the second law defines equilibrium in the capital market. The first law follows from assuming competitive factor markets and constant returns-to-scale. And, in particular, these assumptions imply that the cost-of-capital equals the marginal product of capital (MPK). The second law comes from assuming that total saving must equal to total investment for the circular flow of money to be in equilibrium.

These "laws" are very close to being true by definition, and so the plausibility of a macroeconomic model depends on yielding similar "laws" among its results. Nonetheless, the version of these laws found in Piketty's book neglected some technicalities connected to the continuous replacement of capital equipment as an offset to depreciation.[23] Hence, Van Schaik provided some further elaborations that incorporate both these factors into a slightly more general formulation of Piketty's original laws.[24]

The first law is that the share of national income (GVA) going to the owners of capital is a fixed parameter of the economy. Specifically, capital's share of national income, β, is given by:

$$\beta = rK/Y\,, \qquad \text{8-1}$$

where r is the rate of return (or equivalently the cost-of-capital), K is the capital stock, and Y is the economy's output. The rate-of-return (or equivalently the cost-of-capital) is the interest rate plus the depreciation rate: $r = i + \delta$, where i is the interest rate, and δ is the depreciation rate.

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[23] '*Piketty's laws with investment replacement and depreciation*': https://cepr.org/voxeu/columns/pikettys-laws-investment-replacement-and-depreciation

[24] '*On the link between Piketty's laws*': https://www.ifo.de/DocDL/forum1-15-focus2.pdf

As already mentioned, the second law can be derived from the observation that aggregate savings must equal aggregate investment when the circular flow is in equilibrium:

$$(\delta + g)K = sY, \qquad 8\text{-}2$$

where s is the savings rate, g is the economy's growth rate, and δ is the depreciation rate for capital. A slight rearrangement shows that an economy's "capital coefficient", K/Y, is given by:

$$K/Y = s/(\delta + g)\,. \qquad 8\text{-}3$$

In line with the Solow's classic growth model, Piketty assumes that the economy has a fixed savings rate, s, and that the economy grows at the same rate as the workforce. That is, the savings rate stays the same over many decades even if the values of some of other quantities change over time.[25]

Putting these laws together gives a formula connecting capital's share of income, β, the interest rate, i, the savings rate, s, and the economy's growth rate, g:

$$\beta = r.\left(\frac{K}{Y}\right) = (i + \delta).\left(\frac{s}{\delta + g}\right)$$

This formula says that capital's share of GVA (i.e., β) is proportional to product of the economy's cost-of-capital (i.e., $i + \delta$) and the economy's savings rate (i.e., s), but is inversely proportional to the gross investment rate (i.e., $\delta + g$). An important implication of this formula becomes most apparent when it's rewritten as follows:

$$\frac{\beta}{s} = \frac{i + \delta}{\delta + g} \qquad 8\text{-}4$$

Firstly, the left-hand side of this formula is capital's share of national income (GVA), β, divided by the gross saving rate, s. Thus, the ratio of β to s depends on the ratio of $i + \delta$ to $\delta + g$. Secondly, recall that the gross profit from investing in capital is $(i + \delta)K$ and gross investment is $(\delta + g)K$, implying that the net-profit from investment must be $(i - g)K$. Finally, Piketty's formula implies that the gross return from investing in capital will exceed the gross cost of investment when $i > g$. This means that the ability of the owners of capital to sustain positive net profits, in equilibrium, depends on society keeping the growth rate of the workforce lower than the interest rate.

Why are these positive net profits required for the system to function? Well, it's because people prefer present consumption to future consumption, and so some compensation is needed to induce investors to defer part of their consumption. Moreover, such deferment has a cost because happiness tomorrow is worth a little less than happiness today. In short, people are inherently impatient and so discount the future relative to present. Hence, the positive compensation required by investors reflects what economists call the "time value of money", which for society is given by $i - g$. Note that this concept features prominently in HMT's Green Book, where it is used to weigh the present costs of an investment against the flow of future benefits.

The owners of capital make a living through buying capital and then renting it out. Moreover, the viability of making a living in this way depends on sustaining a positive net-profit. Therefore, the functioning of our economic system depends on the interest rate exceeding the growth rate of the workforce. Thus, the engine of capital accumulation may begin to stutter if there's a long period of ultra-low interest rates.

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[25] Piketty shows that the savings rate takes much the same value in all the major developed economies over long periods of time. This fixed saving rate is his strongest claim, with the rest of his thesis being true almost by definition.

Piketty's "laws" also have important implications for how the economy performs from the perspective of a worker who doesn't own that much capital. This helps us to better understand "what good looks like" in terms of configurations favourable to workers. A takeaway message from this analysis is that the long-run growth rate for GVA is a good measure of the economy's performance, because higher growth should lead to broad-based prosperity for the whole population. That is, higher growth leads to higher incomes for everyone, particularly, for those who don't own much pre-existing wealth, and so earn an income by selling their labour.

Thus, this analysis offers some high-level criteria for judging the attractiveness of alternative equilibria. Such criteria are useful when using comparative statics to explore the effect of changes to the economy's basic parameters on the performance of the real economy in terms of its effect on the earning power (prosperity) of citizens.

8.1 A TRIPLET OF EQUATIONS

Piketty's formula can be written in terms of the rental rate for capital items when the economy is in equilibrium:

$$r = \frac{\beta}{s}(\delta + g)$$

Moreover, this formula can be generated by combining the following pair of fundamental equations:

$$\frac{k}{y} = \frac{\beta}{r}$$

and

$$sy = (\delta + g)k$$

The first equation gives the economy's capital ratio, and the second equation comes from the requirement that savings must equal investment for the system to be in equilibrium. Notice that this set of equations form a triplet: any two equations yield the third equation as an immediate result.

9 THE SOLOW GROWTH MODEL

This section sets up and solves the Solow model. It ends by showing that the Solow model yields a version of Piketty's formula as one of its results.

9.1 THE PASSAGE OF TIME

In this macroeconomic model, time is measured in units of years, and so the evolution of the economy plays out over decades. A year is long enough for savings to be invested in the capital stock but not so long that the capital stock can change dramatically from one moment to the next, which is why it's modelled as a stock variable. It follows that parameters are annualised figures and the same is true of quantities like per capita output.

9.2 NO TECHNOLOGICAL CHANGE

In the very long run, the UK economy will benefit significantly from exciting innovations originating from its science-base, but the economic effects of this kind of technological change occur at a gradual pace. So, to simplify the analysis, it is assumed that there is no significant technological change in the economy due to scientific advances. Firstly, this should be regarded as just another simplifying assumption, much like assuming full employment. (That is, it is a convenient simplification that isn't exactly true.) Secondly, over the period considered in this study, growth in the UK's TFP had been at a historically low level and, as such, it can be conveniently omitted from our model without effecting the parameter estimates.

Understanding why growth in the UK's TFP declined after the *Great Recession* - and still hasn't fully recovered - is clearly of high importance, but it isn't the subject of this paper. So, in the absence of other sources of economic growth, long-run GDP growth depends on the growth rate of the workforce and capital accumulation.

9.3 GROWTH IN THE WORKFORCE

The Solow model assumes full employment, so that the workforce grows as the population grows. It will be shown that, in the steady state, the growth rate of the workforce is the determinant of the economy's long-run growth rate.

Let the first derivative of the size of the economy's workforce with respect to time be denoted as follows: $\dot{L} = dL/dt$. Here, Newton's "dot" notation is being used to denote the first derivative with respect to time – the change in the size of the workforce that occurs between time *t* and time *t* + *dt*, where '*dt*' represents a small interval of time. The workforce grows at a constant proportional rate, g, so that $\dot{L}/L = g$. And, from this, it can be shown that $L = L_0 \exp(gt)$, where L_0 is the initial size of the workforce. That is, L_0 is the size of the workforce at time t = 0.[26]

Note that g is taken to be a basic parameter of our society, unaffected by economic circumstances. Providing that entry into the labour market exceeds the retirement rate, there will be growth in the workforce ($g > 0$).

Based on employment data for non-financial businesses, the size of the working population grew at an average of 1.5% per annum between 2015 and 2019. So, this estimate of the average annual growth rate implies that $g = 1.5\%$. (Own calculations using ONS data.)

Lastly, it will be seen that g is an important parameter of the system, determining not only its trajectory but also the stability of the system's fixed point. That is, a higher value of g makes the equilibrium more robust to small changes in the other parameters.

The economy's output (*Y*) is generated through a process that uses the factors of production: labour (*L*) and capital (*K*). Suppose that aggregate output from the real economy is characterised by a Cobb-Douglas production function with constant returns-to-scale. That is, aggregate production is $Y = AL^{\alpha}K^{\beta}$, where *Y*, *K*, and *L* are aggregate output, capital, and labour, respectively. As discussed, constant returns-to-scale means that if labour and capital were both to double, then so would the economy's output, and this condition requires that $\alpha + \beta = 1$.

Let $y = Y/L$ and $k = K/L$ denote per capita output and capital intensity, respectively. Assuming constant returns-to-scale means that the production function can be written in per capita terms as: $y = f(k)$, where $f(k) = Ak^{\beta}$. Recall that labour productivity, y, is an increasing function of capital intensity, k.

As already discussed, it can be shown that: $f_k(k)k = \beta f(k)$. The LHS (Lefthand Side) is the per capita cost of renting the economy's capital and the RHS (Righthand Side) is the share of the economy's per capita output that goes to owners of capital. Notice that this equation is a version of Piketty's first law of capitalism (defining the economy's capital coefficient).

9.4 THE STATE EQUATION FOR CAPITAL

Capital accumulation depends on the in-flow of new capital from investment exceeding the out-flow of capital that expires due to depreciation. Suppose that each year a set fraction of the capital stock dies when incidents cause machines to become irreparably broken. Let $\delta \in (0,1)$ denote the yearly depreciation rate. For reasons that will become clear in later sections, it is convenient to assume that older machines are no more likely to suffer such accidents

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[26] If half the population is in the workforce, then the population is 2L. The factor of 2 does not affect the fundamentals of the analysis. The implicit assumption is that the dependency ratio remains the same over a long period of time.

than younger machines, so that the machines that expire at the end of the year are representative of the general capital stock. This should be regarded as another simplifying assumption.

Since depreciation occurs at a rate δ, the change in the capital stock is $\dot{K} = Y - C - \delta K$ where C is aggregate consumption. This assumes that capital is augmented by all the output that's not consumed or lost to depreciation. Using $y = f(k)$, the per capita version of this equation becomes:

$$\dot{k} = f(k) - c - (\delta + g)k, \qquad 9\text{-}1$$

where c is per capita consumption.

Proof: Firstly, from the chain-rule for differentiation and the definition of k we get the following identity:

$$\dot{K} = \frac{d}{dt}(kL) = \dot{k}L + \dot{L}k = L\left(\dot{k} + \frac{\dot{L}}{L}k\right)$$

Recall that: $g = \dot{L}/L$. So, combining this identity with the definition of g gives $\dot{K} = L(\dot{k} + gk)$, implying $\dot{K}/L = \dot{k} + gk$. Secondly, starting from the original equation for the evolution of $\dot{K}$ and dividing both sides through by *L* gives us $\dot{K}/L = y - c - \delta k$. Therefore, we must have $\dot{k} + gk = y - c - \delta k$, which gives the main result for $\dot{k}$ on substituting $f(k)$ for y. ■

9.5 SOLOW MODEL

Let s denote the long-run savings rate and suppose this to be a basic parameter of the economy that depends on the parsimoniousness of the nation's citizens, meaning that it's set by people's psychology. From a statistical perspective, the economy's savings rate is its gross investment (Gross Fixed Capital Formation) as a percentage of its aggregate output (GVA). A plausible value for the savings rate was found to be $s = 20\%$. (Own calculations using ONS data.)

The Solow model assumes that a constant faction, s, of aggregate output is always saved, whilst the remaining fraction, $1 - s$, is consumed. Thus, for a given output per capita, $f(k)$, the savings rate determines per capita consumption, c. Hence, per capita consumption is given by $c = (1 - s)f(k)$, so that the per capita investment becomes $c - f(k) = sf(k)$, which then implies that net investment per capita is $\dot{k} = sf(k) - (\delta + g)k$. Note that this equation is closely related to Piketty's second law of capitalism: setting $\dot{k} = 0$ implies that gross investment equals the gross spending on capital items.

In equilibrium, the gross investment rate is $\delta + g$, where δ is the depreciation rate and g is the growth rate for the workforce. Since $\delta = 4.8\%$ and $g = 1.5\%$, the gross investment rate becomes 6.3%.

When characterising the fixed-point of the system, it's helpful to consider the expression on the right-hand side (RHS) of the capital equation: $G(k) := sf(k) - (\delta + g)k$. It can be shown that $G(k)$ is a concave function of k and that it has two roots: $k = 0$ and $k = k^*$, where $k^* > 0$.

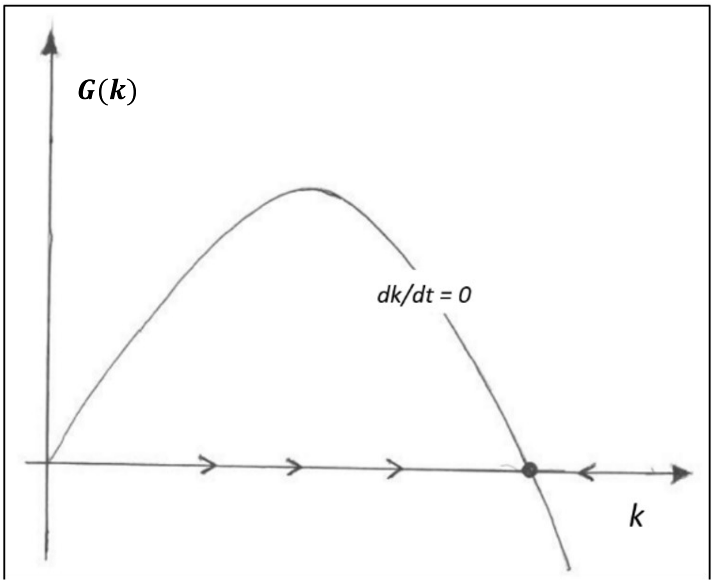

Figure 1: Phase diagram for k

The phase diagram above shows that there exists a unique, equilibrium value of k that is defined by the following expression: $sf(k^*) = (\delta + g)k^*$. Solving this equation for k^* gives us an expression for the steady state value of the capital intensity:

$$k^* = \left(\frac{sA}{\delta + g}\right)^{1/\alpha} \qquad \text{9-2}$$

And, the output per capita becomes:

$$f(k^*) = A.\left(\frac{sA}{\delta + g}\right)^{\beta/\alpha} \qquad \text{9-3}$$

It follows from the phase diagram that k^* is a "sink" (a stable state), which implies that $k \rightarrow k^*$ as $t \rightarrow \infty$, providing that $k_0 > 0$ so that there is some capital in the first place. So, in the long run, as $t \rightarrow \infty$, the capital intensity, K/L, must converge to a constant value, k^*. Since L was assumed to grow at a rate of g in the long-run, K must grow at the same geometric rate, and because of constant returns-to-scale, so does Y and C. Thus, the economy settles into a state where it grows at the same rate as the workforce.

9.6 RECOVERING A VERSION OF PIKETTY'S FORMULA

A version of Piketty's formula can be found among the results of the Solow model. Firstly, it has already been shown that combining a Cobb-Douglas production function with constant returns-to-scale leads to the following equation: $\beta f(k^*) = f_k(k^*)k^*$. The LHS is the share of the economy's per capita output that goes to owners of capital and the RHS is the per capita cost of renting the economy's capital. Secondly, in equilibrium, it must be the case that $\dot{k} = 0$, which was shown to imply that gross savings equal the gross cost of investment:

$$sf(k^*) = (\delta + g)k^*.$$

Combining these results implies that:

$$\frac{\delta + g}{s} = \frac{f(k^*)}{k^*} = \frac{f_k(k^*)}{\beta}$$

From this expression we get a recognisable version of Piketty's formula:

$$\frac{\beta}{s} = \frac{f_k(k^*)}{\delta + g} \qquad \text{9-4}$$

This implies that that, in the Solow model, the cost-of-capital equals the marginal product of capital, $f_k(k^*)$. And, the gross investment rate is given by $\delta + g,$ meaning that the population's growth rate, $g,$ ultimately, sets the growth rate for the economy.

This concludes Part 1 of the report. The following sections set up new elements that will be added to the classic Solow model, to yield a model for a more realistic economy. Namely, one with imperfect production processes and that also accounts for the contribution of the national quality infrastructure.

10 EMPLOYING ENGINEERS TO SUPERVISE PRODUCTION

This section begins Part 2 of this report, which focusses on introducing new elements into the model to account for the effects of imperfect production processes and conformance testing activities. Specifically, Part 2 introduces a new state equation for the reliability of the production process. Along with the existing state equation for capital intensity, this yields a system of two coupled differential equations for the dynamics of the economy. However, the next step is to incorporate the CT engineers into the model so that they can supervise production processes.

The Solow model assumed a perfect production process in which nothing ever goes wrong. However, a more realistic model would allow for a production process that wasn't fully reliable, necessitating the employment of engineers to find, and fix, the malfunctioning machines.

This section introduces a series of new parameters for the employment of engineers and the efficiency of their conformance testing activities. These new parameters can be split into two classes:

- Parameters that are potentially observable and for which we can make some reasonable estimates.
- Unobservable parameters that have theoretical meaning but would be extremely difficult to directly measure, although, common sense suggests a plausible range.

In both cases, the status of these parameters is different from the well-established macroeconomic parameters that feature in the Solow model. That is, the values that appear in this section are reasonable estimates or ball-park numbers. It will require dedicated surveys to fully determine such values, and so this will become a topic of future empirical work.

10.1 THE NEED TO DETECT PROBLEMS IN PRODUCTION

Suppose that the production process isn't fully reliable, so that it is generating a mixture of usable output ("good") and defective output ("bad"). The defective portion of the economy's output is useless, and so must be scrapped. However, its defective nature doesn't become apparent until it reaches a buyer who tries in vain to consume it. That is, such output isn't obviously defective, but a buyer will soon discover that it's unusable, whereupon it will be returned to the seller by a customer wanting a replacement or a refund.

Unless the malfunctions are detected and corrected, the machines that aren't working correctly will continue to produce defective outputs. It follows that, without effective quality control, a growing fraction of output will become defective.

Consequently, engineers must be employed to supervise production and ensure that the machines are working correctly. Employing engineers to supervise production means that any machines that aren't working correctly can be found and then fixed.

10.2 THE EMPLOYMENT OF ENGINEERS

There exists a large pool of 'engineers' capable of supervising production. Let the term 'engineers' be a shorthand for '*STEM professionals employed to do conformance testing*'.

Note that conformance testing is not directly productive, rather it provides confidence that what has been produced meets any regulations and is of dependable quality. So, in an important sense, conformance testing takes place outside the real economy, much like employing a police force and judiciary to ensure “good order”. The people employed in such jobs create something useful to society (confidence), but it cannot be bought and sold in the same way as goods. Therefore, conformance testing diverts resources that - in an unrealistically perfect world - would otherwise be used for production. In other words, conformance testing benefits society (by ensuring that production processes are reliable) but it comes at a cost, because it requires resources that would otherwise be deployed elsewhere.

Employing engineers to supervise production means that any machines that aren’t working correctly can be found, and then fixed, so that they once again work properly. Let E denote the number of CT engineers employed in the economy, so that the number of CT engineers per worker is given by:

$$\text{engineering intensity of the workforce} = e = E/L\,, \qquad \text{10-1}$$

where L is the total size of the workforce. An estimate of e was made using employment data collected by the ONS back in 2017.[27] The details were as follows:

CT Engineers as a Percentage of the Workforce: In 2017, total employment in the UK was estimated to be 31.9 million, implying that $L = 31.9$ million.

A list of occupations that involve making measurements was compiled from the description of the Standard Occupation Codes (SOCs) used by the ONS. Based on this, 3.9% of employment was comprised of people in occupations that require them to make scientific or engineering measurements on a regular basis.

The 1.24 million measurement jobs (in 2017) can be split into three groups:

1. 160 thousand calibration jobs account for 0.5% of employment.
2. 380 thousand testing and analysis jobs account for 1.2% of employment.
3. 700 thousand measurement intensive jobs account for 2.2% of employment.

Conformance testing does not account for all forms of measurement activity. We can regard the first two groups (calibration and testing) as mostly being associated with conformance testing. But only part of the third group (measurement intensive jobs) will connect to conformance testing. For simplicity, we take 50% of this third group and allocated it to conformance testing. This gives 890 thousand people employed in conformance testing, which implies that $E = 890$ thousand.

So, to summarise, our estimates are $L = 31.9$ million and $E = 890$ thousand, from which it follows that $e = 2.8\%$. (Own calculations using ONS data for 2017.)

10.3 THE WAGE EARNED BY AN ENGINEER

Suppose that there’s a deep and fluid labour market for engineers and let ω denote the wage they command in this market.

[27] Based on data coming from respondents to the NMS survey, conformance testing accounts for 63% of what these businesses spent on measurement (testing and analysis). Since standards and regulations are particularly important to businesses using the NMS labs, conformance testing probably accounts for a smaller proportion of the measurement activity undertaken by the general population of businesses in the economy.

If an engineer isn't being paid the going rate, then they can easily take a job with another employer who has a more secure grasp of realities in the labour market.[28] Furthermore, suppose that engineers are highly mobile ("citizens of the world"), so that engineering services are bought and sold in a global market. Consequently, the wage rate for engineers is a parameter that's determined by factors outside our model.

Since E is the number of engineers and ω is their wage, it follows that total spending on conformance testing is given by:

$$\text{spending on conformance testing} = \omega E \qquad \text{10-2}$$

An estimate of this spending has been made using employment data collected by the ONS. The findings were as follows:

Estimating the Wage Rate: In 2017, £40.8 billion was paid to 1.24 million people in occupations that involve making scientific or engineering measurements. Hence, those employed in such occupations earned an average wage of £33 thousand.

An analysis of the wage data (using Standard Occupation Codes) found that this £40.8 billion in wages can be broken down as follows:

1. 160 thousand calibration jobs contribute £4.3 billion in wages.
2. 380 thousand testing and analysis jobs contributes £12.5 billion in wages.
3. 700 thousand measurement intensive jobs contribute £24 billion in wages.

As already discussed, the first two groups are taken to be strongly associated with conformance testing, along with 50% of the third group. So, for employment in conformance testing, this gives us 890 thousand jobs and £28.8 billion in wages.

Our estimates imply that $\omega E = £28.8$ billion. And, as $E = 890$ thousand, it follows that $\omega = £32.4$ thousand. (Own calculations using ONS data for 2017.)

10.4 AN ENGINEER'S "PORTION" OF THE CAPITAL STOCK

Suppose that the capital stock is evenly divided amongst all the CT engineers employed to supervise production. These "portions" provide a convenient way of quantising the capital stock into a great many discreet units, which can be thought of as production plants.

The "portion size" governs the employment of engineers in the same way that the maximum permissible "class size" governs the number of teachers employed by schools. And, for now, suppose that the "portion size" is set by exogenous factors outside the model, much like the savings rate. That is, the "portion size" reflects the preferences of society and is somewhat analogous to class-sizes in schools.

Let ρ denote an engineer's "portion size". Because the average amount of capital under the control of each engineer is k/e, the value of ρ can be inferred from the values of e and k:

$$\text{portion size} = \rho = k/e \qquad \text{10-3}$$

[28] Suppose that neither employers nor governments, have any control over the wage rate for engineers: Any misguided attempts to pay below the market rate precipitates a general exodus of talent.

Moreover, we have already established that $e = 2.8\%$, and so it remains to find an estimate of the capital intensity, k. Note that, in this study, the capital stock is composed of machinery and equipment and does not include assets in the form of land and property.

Capital intensity is the amount of capital per worker in the economy, which has been estimated using ONS data for 2017 and the findings were as follows:

Estimating the Capital Intensity: The UK's stock of assets (wealth) divides into three classes: (1) land; (2) produced assets; and (3) net financial assets. Since claims and liabilities tend to almost cancel out, wealth is mostly divided between land and produced assets. However, in the context of this study, it is only the stock of produced assets that matters as it contains productive capital, such as, plant and machinery. (Land now accounts for about half the UK's stock of wealth but is excluded from our analysis.) The UK's stock of produced assets was worth £4.5 trillion in 2017 according to the ONS's National Balance Sheet. Given that the capital stock of produced assets is mostly comprised of productive assets, this implies that the UK's total capital stock was valued at $£4.5$ trillion. Since total employment in 2017 was 31.9 million, the capital per worker was £141.1 thousand. Hence, our estimates imply that $k = £141.1$ thousand. (Own calculations using ONS data for 2017).

As already discussed, it is possible to find the value of ρ from estimates of e and k: If the UK had a capital intensity of £141.1 thousand and the engineers employed in conformance testing made up 2.8% of total employment, then each engineer would be responsible for supervising machinery worth £5.1 million. Hence, our analysis implies that $\rho = £5.1$ million, and this "portion size" reflects the stringency of the inspection regime. Moreover, it's important to note that ρ is generally to be treated as a fixed parameter of the model: It does not vary over time, and it is unaffected by changes in the size of the capital stock. (If the capital stock increases, then the economy will need to employ more engineers. This is much the same as if the number of children increases, then society will need to employ more teachers.)

In most of this study ρ is treated as a fixed parameter, that is determined by exogenous factors beyond the model. However, in the final sections of this study, we allow ρ to be actively chosen by society by setting the frequency of inspections to maximise the output per capita in the economy's equilibrium. But, for now, treat ρ as if it were just a fixed parameter of the system.

10.5 THE PACE OF INSPECTIONS (SPAN OF CONTROL)

The number of engineers overseeing production won't be so numerous that it's possible to have all machines watched 24 hours a day, for 7 days a week. Rather, an engineer will cycle round a production plant, checking one machine, then another machine, and so on.[29] Nonetheless, given enough time, these engineers are expected to have combed through the whole capital stock.

The pace of inspections is determined by an engineer's span of control. Let a denote the maximum amount of capital that an engineer can reliably supervise. In other words, this is

[29] The situation is like a model for crime detection, where a police officer has a 'beat' that they cycle round. Just as it's not possible to eliminate crime entirely - as the cost would be too exorbitant - it's also not possible to completely eradicate defective outputs.

the maximum amount of capital (in millions of pounds) that an engineer can reliably inspect each year. Hence, a is measured in units of millions of pounds of capital per annum.

These inspections are "rigorous" in the sense that any machine that passes the tests is almost certain to be working correctly, and so producing usable output. As will be discussed in more detail below, a test is "rigorous" if it is 99.9% certain to reject a defective output.

Let us refer to a as the "pace" at which reliable inspections can be performed. The value of this parameter reflects the power of the infra-technology underpinning the National Quality Infrastructure (NQI). Improvements in the infra-technology yield an increase in a, whereas any deterioration in this technology would lead to a decrease in a.

Notice that a provides us with an index for the productivity of CT engineers. Although, a isn't directly estimable, common sense suggests a ballpark range: By the class-size analogy, we'd expect a to be a little bigger than ρ. (If class-sizes were much smaller than a teacher's span-of-control, then society could reduce the number of teachers whilst maintaining educational standards. If class-sizes where much larger than a teacher's span-of-control, then educational standards would be likely to suffer.)

10.6 THE FREQUENCY OF INSPECTIONS

Let n denote the average number of times that a capital item is inspected each year. In other words, n is the typical number of times that the output, coming from a given machine, will be inspected before it enters the market. If the number of engineers employed in the economy is E and a is the "pace of inspections", then the frequency with which the whole stock of machines can be inspected is aE/K. (This is the number of complete sweeps performed by the engineers each year.) It follows from the definitions of the "portion size", ρ, and the "pace of inspections", a, that an expression for the yearly frequency of these inspections, n, is as follows:[30]

$$\text{frequency of inspections} = n = a/\rho \qquad \text{10-4}$$

Furthermore, the amount of time that elapses between successive inspections (as a proportion of a year) is given by ρ/a. That is, if n is the frequency of inspections, then n must be the interval of time between inspections.

10.7 THE COST OF SUPERVISION

The ongoing cost of inspections is born by the businesses using capital items for production, which resembles the yearly cost of MOTs for cars. τ denotes the yearly cost of employing CT engineers to supervise one million pounds worth of capital. Hence, τ can be thought of as the cost of employing CT engineers to supervise capital equipment as a percentage of its worth, and so resembles a service charge or a tax.

Since the amount of capital is based on its worth (and measured in millions of pounds), it follows that τ is the cost of the engineers needed to supervise a million pounds worth of capital. The cost of CT engineers per million pounds of capital is given by:

$$\text{supervision cost} = \tau = (\omega E)/K\,, \qquad \text{10-5}$$

where ω is an engineer's wage, E is the number of engineers employed in conformance testing, and K is the value of the economy's capital stock. The numerator of this expression is the cost of conformance testing (ωE), and the denominator is the size of the capital stock (K).

[30] This is rather like the equation in Physics for the speed of a wave: speed = frequence × wavelength.

Notice that the supervision cost can be written as $\tau = \omega \times (E/K)$. By inverting the basic equation for ρ, we have $E/K = 1/\rho$. Hence, the yearly cost of supervising one million pounds of capital can be written as:

$$\text{supervision cost} = \tau = \omega/\rho \qquad 10\text{-}6$$

Moreover, it has already been established that $\omega = £32.4$ thousand and $\rho = £5.1$ million. Evaluating our formula for τ using these estimates of ω and ρ gives:

$$\tau = \frac{£32.4 \text{ thousand}}{£5.1 \text{ million}} = 0.64\%$$

In a subsequent section of this report, it will be shown that τ (along with the savings rate) determines the level of spending on conformance testing as a proportion of the economy's GVA. Lastly, since $n = a/\rho$ and $\tau = \omega/\rho$, it follows that:

$$\tau = \frac{\omega}{a} n. \qquad 10\text{-}7$$

As already discussed, it's reasonable to suppose that society has very little control over the wage of CT engineers, ω, or the pace of testing, a. Hence, this formula implies that the supervision cost, τ, is proportional to the frequency of inspections, n.

10.8 UPDATING AND MAINTAINING THE INFRA-TECHNOLOGY

Standardisation plays an essential role in codifying the routines on which the division-of-labour is based, and so helps to breakdown a complex process into a series of simpler sub-processes. Let us assume that the capital stock is composed of many different types of machinery, and that each type is associated with a particular sub-process. Standardisation allows production to be organised such that each engineer oversees one part of the process. This promotes division-of-labour, and combined with learning-by-doing, this leads to specialisation. Hence, the greater the availability of standards, the greater the scope for specialisation, which in turn yields greater economies-of-scale and thus higher productivity.

Standards also reduce transaction costs, and thereby help to grow the volume of business-to-business transactions. This leads companies to buy-in commoditised components and to outsource their more routine operations, thereby shrinking the scope and complexity of what they need to manage inhouse. By this means, standards help to reduce the boundaries of the firm, and so underpins further economies-of-scale that are achieved through enabling structural changes in the economy.

Many technical standards are associated with measurement and testing. Choudhary (2013) found that a large percentage of standards make detailed references to measurement and testing procedures:

- One quarter of standards include both a reference to a test procedure and measurement ('narrow' count of measurement standards)
- Two-thirds contain either or both terms (the 'broad' count of measurement standards)

The pace of testing measures the productivity of the CT engineers, which will be influenced by the information content of technical standards. Hence, the pace of testing will depend on the public investments in the infra-technology that underpins the National Quality Infrastructure (NQI).

A more concrete conception of the information content of standards can be arrived at by seeing the standards as "manuals", describing best practice with regards to the operation and maintenance of machinery and equipment. That is, standards help the real economy to make the best use of the capital items at its disposal. From which it follows that there will be a pairing between types of capital item and standards. This implies that there is an overlay between the capital stock and the stock of standards, in the sense that they are coextensive.

However, the capital stock is constantly being refreshed, because new capital items are added and old capital items exit due to depreciation, and this necessitates a corresponding refreshing of the standards stock. This is because the capital items entering the stock won't be quite the same as those leaving it. Moreover, as fundamentally new species of capital item are added to the capital stock, new standards will be needed to define best practice.

This means that unless regular efforts are made to update the stock of standards, the effectiveness of the infra-technology will decline as it starts to lose its relevance. Therefore, if it weren't for the in-flow of new and updated standards, the information content of standards is liable to decline at the same rate as the capital stock is being refreshed.

In equilibrium, the gross investment rate determines the rate at which the capital stock is refreshed. Consequently, it's this gross investment rate that determines the rate at which stock of standards loses its information content. Thus, without the efforts to update and refresh the stock of standards, we should expect its information content to decline at a rate of 6.3% a year. Therefore, to prevent this happening, the relevance of the stock of standards is maintained through the countervailing efforts of the NMS laboratories (amongst others) who work to update and refresh its technical content.

11 MODELLING AN IMPERFECT PRODUCTION PROCESS

This section sets up the model for production in the realistic situation where production processes aren't fully reliable.

11.1 THE PROPORTION OF MACHINES WORKING CORRECTLY

Suppose that a newly installed machine (whose purchase was financed through investment) will always be set-up correctly, and so such a machine is initially guaranteed to produce usable output. However, as time goes on, there will be a growing possibility that something has gone wrong so that it's no longer functioning correctly and thus needs to be reset.

As already discussed, the capital stock is divided amongst the CT engineers into $E(t)$ "portions", such that there is a one-to-one pairing between "portions" and CT engineers. Let $v(t)$ denote the fraction of "portions" containing machines that are functioning correctly. At time *t*, the fraction of "portions" in which every machine is functioning correctly is given by:

$$v(t) = \Pr(\text{good at } t). \qquad 11\text{-}1$$

At time *t*, the fraction of "portions" containing at least some machines that are malfunctioning is given by:

$$1 - v(t) = \Pr(\text{bad at } t). \qquad 11\text{-}2$$

We will refer to $v(t)$ as the "reliability" of the production process at time *t*. Since the capital stock is composed of a great many such "portions", $v(t)$ tracks the proportion of the capital stock that is functioning correctly.

11.2 THE TRANSITION RATE

At any instant in time, a machine is in one of two possible states: either it is producing outputs that conform to a given specification ("good") or it has malfunctioned and is producing defective outputs ("bad"). Suppose that at any instant of time there's some probability that a previously "good" machine begins to malfunction (so that it goes from "good" to "bad").

Let $\varepsilon \in (0,1)$ denote the proportion of "good" machines that go "bad" during a year, and this will be referred to as the "transition rate":

$$\varepsilon.dt = \Pr(\text{bad at } t + dt \mid \text{good at } t). \qquad 11\text{-}3$$

It can be seen that:

$$\Pr(\text{bad at } t + dt) - \Pr(\text{bad at } t) = \Pr(\text{bad at } t + dt \mid \text{good at } t) \times \Pr(\text{good at } t),$$

Moreover, this implies that:

$$\Pr(\text{bad at } t + dt) - \Pr(\text{bad at } t) = \varepsilon.dt \times \Pr(\text{good at } t).$$

As already discussed, $v(t)$ denotes the proportion of machines that are functioning correctly. So, in terms of this notation, the proportion of machines that go "bad" between t and t + dt is given by:

$$\Pr(\text{bad at } t + dt) - \Pr(\text{bad at } t) = \varepsilon.dt \times v(t).$$

Considering the change in $v(t)$ with respect to time (using Newton's dot notation) leads us to the following expression:

$$\dot{v}(t)dt = \Pr(\text{good at } t + dt) - \Pr(\text{good at } t) = -[\Pr(\text{bad at } t + dt) - \Pr(\text{bad at } t)]$$

In terms of this notation, the expression above can be rewritten as:

$$\frac{dv}{dt} = \dot{v}(t) = -\varepsilon v(t).$$

However, this simple equation only holds when there is no investment and no conformance testing, which requires: $\dot{K}(t) + \delta K(t) = 0$ and $E(t) = 0$.

Since installing brand new machines and/or resetting the malfunctioning machines creates a tendency for $v(t)$ to increase, the general equation for $\dot{v}(t)$ becomes:

$$\frac{dv}{dt} = \dot{v}(t) = \text{change at } t \text{ due to positive influences} - \varepsilon v(t). \qquad 11\text{-}4$$

Subsequent sections are devoted to finding formulae for the effect of installing new machines and the effect resetting malfunctioning machines. Combining these formulae with the expression above will yield a general equation for $\dot{v}(t)$.

11.3 THE PRODUCTION OF USABLE OUTPUT

Suppose that some fraction of the machinery, within the capital stock, has begun to malfunction, meaning that it has started to produce defective output. Thus, it's as if the economy has lost a little of its capital stock: $K \to vK$, where $v \in (0,1)$. Next, if workers are unaware that some of the machines are malfunctioning, and as labour and capital are always used in fixed proportions, it follows that the same faction of labour will also have ceased to be productive. Again, it's as if the economy has lost a little of its labour, so that $L \to vL$, where $v \in (0,1)$.

If v represents the proportion of resources committed to productive work, then usable output is $F(vL, vK) = A(vL)^{\alpha}(vK)^{\beta}$. Finally, constant returns-to-scale ($\alpha + \beta = 1$) implies that usable output is the proportion of machines that are working properly multiplied by maximum possible output in the extremely unlikely situation where all machines are working correctly:

$$F(vL, vK) = v \times F(L, K)$$

The parameter A now sets the economy's potential TFP, but this maximum is only attainable in the extremely unlikely situation where production is fully reliable ($v = 1$). Hence, it is helpful to think of A as the maximum attainable TFP with the economy's existing technology. As already discussed, the effective TFP will be less than the maximum attainable TFP.

If there were no type-1 errors (false positives) and testing were costless, then the per capita output available for either consumption or investment would be $vf(k)$, where the maximum possible per capita output is $f(k) = Ak^{\beta}$. Notice that v is the proportion of this potential output that is actually usable.

Finally, the inclusion of v in our expression implies that output (and thus investment) now depends on the proportion of machines that are functioning correctly.

12 MODELLING THE CONFORMANCE TESTING PROCESS

The change in reliability, $\dot{v}(t)$, will depend on both $v(t)$ itself and the degree to which the capital stock is being supervised by CT engineers. The "scrap rate" and the "rebate rate" are closely connected to conformance testing and determine the reliability of the production process. Let these quantities be defined as follows:

The "scrap rate" is made up of two components, the first being the loss from type-1 errors and the second being the defective output that is correctly scrapped before it reaches customers. Hence, the "scrap rate" is the portion of total output that is either (necessarily) scrapped because it is defective or is (regrettably) scrapped due to type-1 errors in the testing process. The first component represents unavoidable losses come with an imperfect production process,[31] whereas the second component relates to the portion of scrapped output that could be eliminated if there were no type-1 errors.

The "rebate rate" is the portion of total output that is returned to sellers by customers because it's found to be defective. Hence, the "rebate rate" refers to the proportion of sales in which the goods fail during the warranty period, and so are returned to sellers by customers wanting a refund.

The aim of this section is to derive formulae for the "scrap rate" and "rebate rate".

12.1 TYPE-1 AND TYPE-2 ERRORS

Let $y(t)$ denote the per capita revenue generated by selling usable output to customers. Note that this is a little less than the total usable output from production because some of it is consumed by the conformance testing process: The existence of type-1 errors (false positives) means that some of the output is scrapped even though it is perfectly fine.

Suppose that when an engineer encounters a malfunctioning machine there is a high probability that the conformance test correctly tells them that it's gone "bad", and so they scrap its output and reset the machine. It is also possible that when an engineer encounters a properly functioning machine that they make the mistake of resetting it and scrap its output even though it's really working perfectly fine.

The null hypothesis (H_0) is that the products produced by a machine are "good", in the sense that they conform to specification. The alternative hypothesis (H_1) is that the products produced by the machine are "bad", meaning that they are defective. Rejecting the null hypothesis when it is true is a type-1 error (or "false-positive"). Accepting the null hypothesis when the alternative hypothesis is true is a type-2 error (or "false-negative").

A binary variable, $\mathbb{S}$, that denotes the state of the goods produced by a production plant. The goods being assessed by an engineer are either "good" ($\mathbb{S} = 0$) or "bad" ($\mathbb{S} = 1$). $\mathbb{X}$ is a binary variable denoting the outcome of the conformance test. During an inspection, an engineer either accepts the good ($\mathbb{X} = 0$) because it passes the test or rejects the good ($\mathbb{X} = 1$) because it fails the test.

The table below gives the conditional probability of accepting or rejecting under each scenario, where the columns give the state (good or bad) and the rows give the outcome of the test (pass or fail). Notice that the probabilities in the columns must sum to unity.

[31] These "unavoidable losses" relate to the output from malfunctioning machines, and the role of the NQI is to try and detect this defective output before it enters the supply chain.

Table 2: Contingency Table of Conditional Probabilities

	Good: $\mathbb{S} = 0$	Bad: $\mathbb{S} = 1$
Pass: $\mathbb{X} = 0$	$\wp_{0\|0} = \Pr(\text{ pass} \mid \text{good })$	$\wp_{0\|1} = \Pr(\text{ pass} \mid \text{bad })$
Fail: $\mathbb{X} = 1$	$\wp_{1\|0} = \Pr(\text{ fail} \mid \text{good })$	$\wp_{1\|1} = \Pr(\text{ fail} \mid \text{bad })$

$\wp_{1|0} \in (0,1)$ denotes the probability of type-1 error (or "false-positive"), which occurs when an engineer tests the machines in their portion of the capital stock and mistakenly rejects output that is, in fact, "good". Suppose that when this happens a year's worth of output is lost from the falsely impugned machines under their supervision. Part of this loss comes from recalling suspect output, and part comes from down-time whilst the machines are reset and recertified. The inspection regime will be set up to keep such losses to a minimum.

$\wp_{0|1} \in (0,1)$ denotes the probability of a type-2 error (or "false-negative"), where the tests incorrectly tell an engineer that the machines in their part of the capital stock are working just fine even though, in fact, some of them are malfunctioning and thus producing defective output. It follows that the probability of correctly rejecting the null hypothesis (the statistical power) of a conformance test is given by:

$$\text{statistical power} = \wp_{1|1} = 1 - \wp_{0|1} \,. \qquad 12\text{-}1$$

Let us say that a test is "rigorous" if it is almost certain to reject defective goods. That is, a test should almost never permit the "bad" items to pass through the process undetected. Let us suppose that $\wp_{0|1} = 0.1\%$, which then entails $\wp_{1|1} = 99.9\%$ (i.e., a test with close to the maximum possible statistical power). Hence, a test is said to be "rigorous" if the chance of a type-2 error is only 0.1%.

From the definitions of $\wp_{1|0}$, $\wp_{0|1}$ and $v(t)$, the likelihoods of scrapping a "good" product or accepting a "bad" product will be as follows:

- The likelihood of a scrapping a "good" product is $\wp_{1|0} v(t)$,
- The likelihood of accepting a "bad" product is $\wp_{0|1}[1 - v(t)]$.

Achieving desirably low values of both $\wp_{1|0}$ and $\wp_{0|1}$ depend on having ways to minimise the uncertainty of the measurement process, which involves eliminating sources of systematic error by using precisely calibrated instruments and certified reference materials.

Lastly, sometimes it will be convenient to use a slightly simplified notation: Let $p \equiv \wp_{1|0}$ and $q \equiv \wp_{1|1}$. Using this notation the contingency table is as follows:

Table 3: Contingency Table of Conditional Probabilities

	Good: $\mathbb{S} = 0$	Bad: $\mathbb{S} = 1$
Pass: $\mathbb{X} = 0$	$1 - p = \Pr(\text{ pass} \mid \text{good })$	$1 - q = \Pr(\text{ pass} \mid \text{bad })$
Fail: $\mathbb{X} = 1$	$p = \Pr(\text{ fail} \mid \text{good })$	$q = \Pr(\text{ fail} \mid \text{bad })$

Note that there are only two outcomes (pass or fail), and so $1 - p = \wp_{0|0}$ and $1 - q = \wp_{0|1}$. In other words, p is the likelihood of a false positive; $1 - q$ is the likelihood of a false negative; and q is the statistical power of the test. This is detailed in the contingency table above, where each column sums to unity.

12.2 THE STANDARD DEVIATION OF MEASUREMENT RESULTS

Conformance tests involve making measurements, and these measurements come with an unknown error. But across the whole population of similar measurements, these errors will follow a known distribution. This subsection shows that there is a trade-off between the likelihood of type-1 and type-2 errors, and that the nature of this trade-off depends on the standard deviation of measurement errors.

Suppose that there is some critical characteristic of the parts being produced that is measured during conformance tests. To make the situation more concrete, suppose that this part is a piston shaft, and that the piston-shaft combination only work correctly when the shaft has specified diameter. A practically meaningful deviation from the specified diameter can be set to unity, without loss of generality. (One can imagine that if the dimension of a part is off by more than one unit, then the part is obviously defective, and so a test isn't needed.) This represents the tolerance to which the part must be produced. For example, the diameter of a piston shaft might need to be accurate to the nearest millimetre (mm).

Let D denote the deviation of the critical characteristic from its target value and let $\mathbb{E}[D]$ denote its expected value. Suppose that the production process is either producing parts that conform to specification ($\mathbb{E}[D] = 0$) or it is producing parts whose dimension are off by one unit ($\mathbb{E}[D] = 1$).

The measured value comes with an unknown measurement error, which can be positive or negative:

$$\text{measured value} = \mathbb{E}[D] + \text{error}, \text{ where } \text{error} \sim \mathcal{N}(0, \sigma^2).$$

That is, the 'error' is drawn from a normal distribution that is centred on zero. Let σ be expressed in terms of the practically meaningful deviation. That is, σ is expressed in terms of the size of the deviation that occurs under the hypothesis that the parts are defective (i.e., off by 1mm). Hence, σ can be thought of as the Relative Standard Deviation (RSD) of the measurement process. The standard deviation depends on the expanded uncertainty of the measurement process, incorporating a mixture of random and systematic sources of error.

- Small random errors occur each time a measurement is made due to slight differences in the laboratory conditions, the sample, or the instrument. The effect of such errors can be minimised by taking multiple measurements and then taking an average. The idea being that any random "noise" ought to cancel itself out through the averaging process.
- Systematic errors can't be removed through the averaging of multiple measurements. These are biases that come with using a particular instrument and/or technique. The overall influence of such biases can be gauged through comparing the results from many different labs (using different techniques and instruments) that participate in a proficiency testing scheme.

There are two hypotheses under consideration: the "null hypothesis" is that the parts conform to specification; and the "alternative hypothesis" is that the parts are defective. The null hypothesis can be written as: $\mathrm{H}_0\colon \mathbb{E}[D] = 0 \Leftrightarrow D \sim \mathcal{N}(0, \sigma^2)$. And, the alternative hypothesis can be written as: $\mathrm{H}_1\colon \mathbb{E}[D] = 1 \Leftrightarrow D \sim \mathcal{N}(1, \sigma^2)$.

Let $T = (D - \mathbb{E}[D])/\sigma$ be a standardised version of the measured deviation from the target value.

- Under H_0, $T = D/\sigma \sim \mathcal{N}(0,1)$.
- Under H_1, $T = (D-1)/\sigma \sim \mathcal{N}(0,1)$.

Let $\Phi(\cdot)$ denote the Cumulative Distribution Function (CDF) for a random variable, T, with a standardised normal distribution:

$$\Phi(z) = \Pr(T \le z) = \int_{\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}x^2\right) dx \qquad \text{12-2}$$

Let $\Phi^{-1}(\cdot)$ denote the inverse of this CDF so that $\Phi^{-1}(p) = z$ if and only if $\Phi(z) = p$.

The likelihood of a type-1 error (false positive) is $\wp_{1|0} = \Pr(\text{reject } \mathrm{H}_0|\mathrm{H}_0 \text{ is true})$, and so $\wp_{1|0}$ is the confidence level of a one-sided test (i.e., $T > 0$). Hence, the critical value for the one-sided test is as follows: $\hat{z} = \Phi^{-1}(1 - \wp_{1|0})$, meaning that we reject the null hypothesis if $T > \hat{z}$, where $T = D/\sigma$.

The likelihood of a type-2 error (false negative) is $\wp_{0|1} = \Pr(\text{accept } \mathrm{H}_0|\mathrm{H}_1 \text{ is true})$, and the statistical power of the test is $1 - \wp_{0|1} = \Pr(\text{reject } \mathrm{H}_0|\mathrm{H}_1 \text{ is true})$. The null hypothesis will be rejected when T exceeds the critical value, and so it follows: $1 - \wp_{0|1} = \Pr(T > \hat{z}|\mathrm{H}_1 \text{ is true})$, where $T = D/\sigma$.

Based on this set up, it is possible to derive a formula connecting σ to the likelihood of type-1 errors, $\wp_{1|0}$, and the statistical power of the test, $\wp_{1|1}$:

$$\sigma = \frac{1}{\Phi^{-1}(1 - \wp_{1|0}) - \Phi^{-1}(1 - \wp_{1|1})} \qquad \text{12-3}$$

This formula implies that, for a given a value of σ, there is a tension between the statistical power of the test, $\wp_{1|1}$, and the likelihood of type-1 errors, $\wp_{1|0}$. That is, for a fixed value of σ, $\wp_{1|0}$ increases if $\wp_{1|1}$ increases, and vice versa.

Proof. The statistical power of the test can be written as:

$$1 - \wp_{0|1} = \Pr(T > \hat{z}|\mathbb{E}[D] = 1), \text{ where } T = D/\sigma.$$

Now, focus on the inequality $T > \hat{z}$. Subtracting $1/\sigma$ from both side of the inequality gives:

$$1 - \wp_{0|1} = \Pr[(D - 1)/\sigma > \hat{z} - (1/\sigma)|\mathbb{E}[D] = 1]$$

Under the H_1, $\mathbb{E}[D] = 1$ and $(D - 1)/\sigma \sim \mathcal{N}(0,1)$, which implies that:

$$\Pr[(D - 1)/\sigma \le \hat{z} - (1/\sigma)|\mathbb{E}[D] = 1] = \Phi[\hat{z} - (1/\sigma)].$$

There are two possibilities, either $(D - 1)/\sigma \le \hat{z} - (1/\sigma)$ or $(D - 1)/\sigma > \hat{z} - (1/\sigma)$. Hence, the likelihoods of an event happening or it not happening must sum to unity, which implies that:

$$\Pr[(D - 1)/\sigma > \hat{z} - (1/\sigma)|\mathbb{E}[D] = 1] = 1 - \Phi[\hat{z} - (1/\sigma)].$$

Combining this result with our expression for $1 - \wp_{0|1}$ gives: $1 - \wp_{0|1} = 1 - \Phi[\hat{z} - (1/\sigma)]$. And, this implies that: $\wp_{0|1} = \Phi[\hat{z} - (1/\sigma)]$. Since $\hat{z} = \Phi^{-1}(1 - \wp_{1|0})$, this can be rewritten as: $\wp_{0|1} = \Phi[\Phi^{-1}(1 - \wp_{1|0}) - (1/\sigma)]$. Applying $\Phi^{-1}(\cdot)$ to both sides of this expression yields: $\Phi^{-1}(\wp_{0|1}) = \Phi^{-1}(1 - \wp_{1|0}) - (1/\sigma)$. From this we get the main result after a little rearrangement, whilst recalling that: $\wp_{0|1} = 1 - \wp_{1|1}$. This concludes the proof. ■

13 A POISSON MODEL FOR TYPE-1 AND TYPE-2 ERRORS

This section introduces a Poisson model for the rate of false-positives and false-negatives. The following analysis shows that the rate at which these errors occur depends on the confidence level of the test, its statistical power, and the frequency of the inspections.

13.1 THE REGRET RATE

The output from a machine that's thought to be malfunctioning will be scrapped and the machine itself will then be set up again and recertified. However, due to type-1 errors in the testing process, machines that are working correctly will occasionally be flagged as malfunctioning. In this situation, the process of recertifying the machine is likely to show that the machine was really working fine all along, and this will lead to regret because this implies that its output has been needlessly scrapped.

Let θ denote the proportion of viable output that is regrettably scrapped each year due to type-1 errors. Hence, $1-\theta$ is the proportion of viable output that makes it through the conformance testing process, and so is available for either consumption or investment. Let us refer to θ as the “regret rate”.

θ can be defined more formally using calculus and conditional probabilities:

$$\theta.dt = \Pr(\text{failed at } t + dt | \text{ good at } t) \qquad 13\text{-}1$$

$$(1-\theta).dt = \Pr(\text{passed at } t + dt | \text{ good at } t) \qquad 13\text{-}2$$

Here, ‘failed’ is an abbreviation of ‘*the machine’s output “failed” a conformance test,*’ and ‘good’ is an abbreviation of ‘*the machine is producing genuinely “good” output.*’

In this section, a formula for θ will be derived using a Poisson model for the rate at which machines are reset by the engineers.

The type-1 errors will manifest themselves as lost output, and such loses eat into the surplus that is available for investment: Suppose that every time an engineer encounters a well-functioning machine there is some probability that a conformance test tells them that it’s gone “bad”, so that they scrap its output and reset the machine. Furthermore, suppose that when a portion of machine is wrongly found to be malfunctioning, a year’s worth of production is lost from that portion of machinery. This lost output is partly a consequence of recalling the supposedly defective goods and partly due to the dead-time created whilst the suspect machines are out of action.

Using Bayes Theorem, the likelihood that a type-1 error occurs between time t and time t + dt can be written as follows:

$$\Pr(\text{good \& failed at } t + dt) = \Pr(\text{failed at } t + dt | \text{ good at } t) \times \Pr(\text{good at } t)$$

Since $v(t) = \Pr(\text{good at } t)$, this becomes:

$$\Pr(\text{good \& failed at } t + dt) = \Pr(\text{failed at } t + dt | \text{ good at } t) \times v(t)$$

Turning this into a useful formula requires an expression for the conditional probability of a type-1 error, and it will be shown that this depends on the frequency of inspections, n.

To derive a formula for the likelihood of type-1 errors, it’s helpful to consider what would happen to a prefect machine that somehow never malfunctioned. The likelihood of this special machine being unnecessarily reset by a given engineer will depend on: (i) how frequently the engineers encounter this machine; and (ii) the probability of an engineer making a type-1 error during such an encounter.

As discussed, the expected number of inspections each year (frequency) is denoted by n. Let $p_{1|0} \in (0,1)$ denote the probability of a type-1 error (a “false-positive”) where a conformance test incorrectly tells an engineer that a well-functioning machine is malfunctioning. Hence, the expected number of type-1 errors befalling this machine in a year is simply $p_{1|0}n$.

For a given machine, the yearly number of unnecessary resets, X, roughly follows a Poisson distribution $X \sim \text{Po}[\mathbb{E}(x)]$, where $\mathbb{E}(x) = p_{1|0}n$ is the expected number of unnecessarily resets in a year. Hence, the likelihood of a given machine being rest x times in a year is as follows:

$$\Pr(X = x) = \exp[-\mathbb{E}(x)]\frac{[\mathbb{E}(x)]^x}{x!} = \exp(-p_{1|0}n)\frac{(p_{1|0}n)^x}{x!}$$

(The focus on a prefect machine fits the Poisson model, because it allows the machine to be unnecessarily reset multiple times by the CT engineers.)

The situation in which the machine isn't reset corresponds to $x = 0$. It follows from the formula for a Poisson distribution that the likelihood that none of the CT engineers unnecessarily reset this perfect machine is given by:

$$\Pr(X = 0) = \exp(-p_{1|0}n).$$

However, an ordinary machine - that isn't malfunctioning - will be unnecessarily reset if any one of the engineers incorrectly decides that it has been malfunctioning. That is, a well-functioning machine will be unnecessarily reset if any one of the engineers makes a type-1 error. Hence, the situation in which the machine is reset corresponds to $X > 0$. Furthermore, a machine is either reset during a unit of time or it isn't, which implies that:

$$\Pr(X > 0) = 1 - \Pr(X = 0) = 1 - \exp(-p_{1|0}n).$$

Thus, the probability of a "good" machine being unnecessarily reset becomes:

$$\theta. dt = \Pr(\text{failed at } t + dt | \text{ good at } t) = [1 - \exp(-p_{1|0}n)]. dt$$

Which then implies the following formula for the false-positives rate:

$$\theta = 1 - \exp(-p_{1|0}n). \qquad \text{13-3}$$

Moreover, as has already been shown, the proportion of output lost due to type-1 errors is the conditional probability, given above, multiplied by the proportion of machines that function correctly, $v(t) = \Pr(\text{good at } t)$. This yields the following expression:

$$\Pr(\text{good \& failed at } t + dt) = [1 - \exp(-p_{1|0}n)]v(t). dt$$

Which is then equivalent to:

$$\Pr(\text{good \& failed at } t + dt) = \theta v(t). dt$$

This expression shows how the rate at which "good" output is scrapped depends on both the reliability of production, $v(t)$, and the frequency with which the capital items are being inspected.

13.2 THE DETECTION RATE

Let ϕ denote the proportion of the "bad" machines in the capital stock that are found by the CT engineers (during a year) and then reset so that they then work correctly. In other words, ϕ is the ability of the NQI to find true positives. And, thus, ϕ is a proxy for how effectively the NQI is achieving its principal goal of stopping defective output from reaching the market. Let us refer to ϕ as the "detection rate".

ϕ can be defined using the language of calculus and conditional probabilities:

$$\phi. dt = \Pr(\text{failed at } t + dt | \text{ bad at } t), \qquad \text{13-4}$$

$$(1 - \phi). dt = \Pr(\text{passed at } t + dt | \text{ bad at } t). \qquad \text{13-5}$$

In this section, a formula for ϕ will be derived using a Poisson model for the rate at which machines are reset by the CT engineers.

The reliability of production at time t, $v(t)$, depends partly on the probability of "bad" machines being fixed. Bayes Theorem implies that the flow of correct detections that occurs between time t and t + dt is given by:

$$\Pr(\text{bad \& failed at } t + dt) = \Pr(\text{failed at } t + dt \mid \text{bad at } t) \times \Pr(\text{bad at } t)$$

Using the definition of $v(t)$, this becomes:

$$\Pr(\text{bad \& failed at } t + dt) = \Pr(\text{failed at } t + dt \mid \text{bad at } t) \times [1 - v(t)]$$

The CT engineers search through the machines in the capital stock looking for those that are malfunctioning, and so need to be reset. Before conducting conformance tests on a machine, an engineer doesn't know whether a given machine is malfunctioning or not. Hence, the engineers will encounter these malfunctioning machines at random as they search through the capital stock.

To derive a formula for $\Pr(\text{failed at } t + dt \mid \text{bad at } t)$, consider the likelihood of the engineers finding and fixing one specific machine that's, somehow, almost always malfunctioning. The idea is that, unlike all the other machines, this one is fundamentally faulty, and so immediately flips back to malfunctioning after it's reset.[32]

The likelihood of a particular malfunctioning machine being fixed by an engineer will depend on the frequency of the inspections, as well as the probability that an engineer correctly detects its malfunction during their encounter.

The average number of times this fundamentally faulty machine will be reset in a year is number of inspections each year multiplied by the probability that an engineer decides to reset the machine during an encounter. A formula for this can be found as follows: Firstly, as discussed, the expected number of inspections each year (frequency) is n. Secondly, recall that the likelihood of an engineer correctly detecting a malfunction during an encounter with a faulty machine is $p_{1|1} = 1 - p_{0|1}$. Hence, the expected number of resets in a year is $p_{1|1}n$.

In this situation, the yearly number of resets follows a Poisson distribution $X \sim \text{Po}[\mathbb{E}(x)]$, where $\mathbb{E}(x) = p_{1|1}n$ is the average number of times that this faulty machine is reset in a year. Hence, the likelihood of a given machine being rest x times in a year is as follows:

$$\Pr(X = x) = \exp[-\mathbb{E}(x)]\frac{[\mathbb{E}(x)]^x}{x!} = \exp(-p_{1|1}n)\frac{(p_{1|1}n)^x}{x!}$$

It follows from the formula for a Poisson distribution that the likelihood that none of the engineers manage to find and "fix" our faulty machine is given by:

$$\Pr(X = 0) = \exp(-p_{1|1}n)$$

Since $\Pr(X \geq 1) = 1 - \Pr(X = 0)$, this implies that the likelihood of at least one engineer finding, and then fixing, the malfunctioning machine is given by:

$$\Pr(X \geq 1) = 1 - \exp(-p_{1|1}n)$$

However, an ordinary machine that happens to be malfunctioning (but isn't fundamentally faulty) will be correctly reset if any one of the CT engineers detects that it has been malfunctioning. Hence, the probability of a malfunctioning machine being reset becomes:

$$\phi.dt = \Pr(\text{failed at } t + dt \mid \text{bad at } t) = [1 - \exp(-p_{1|1}n)].dt$$

Which then implies the following formula for the detection rate:

$$\phi = 1 - \exp(-p_{1|1}n). \qquad \text{13-6}$$

[32] The machines that are not fundamentally faulty have some small probability of going from "good" to "bad". Furthermore, the random process governing these transitions is memoryless, in the sense that, assuming a machine is in the "good" state, the likelihood of it flipping into the "bad" state doesn't depend on how long ago it was reset.

The attributable change in $v(t)$ that occurs between time t and t + dt is given by:

$$\Pr(\text{bad \& failed at } t + dt) = [1 - \exp(-\wp_{1|1} n)][1 - v(t)].dt$$

Which is equivalent to:

$$\Pr(\text{bad \& failed at } t + dt) = \phi[1 - v(t)].dt$$

This expression shows how the rate at which "bad" output is rejected depends on both the reliability of production, $v(t)$, and the frequency with which the capital stock is inspected by engineers, n.

14 SCRAPPAGE, REBATES, AND REVENUE

This section introduces some further quantities that are helpful when setting up the model. These include the "scrap rate", the "rebate rate", and the revenue generated from production. Furthermore, these quantities feature in a decomposition of the total output that splits it into three mutually exclusive categories.

14.1 SCRAPPAGE, REBATES, AND SUCCESSES

The "scrap rate" is the proportion of total output that is rejected during the conformance testing process. This rejected output will be a mixture of defective output that's caught before it's sold to a customer and output that isn't defective but nonetheless falls prey to type-1 errors. Hence, the "scrap rate" can be defined as follows:

$$\begin{aligned}(\text{scrap rate}).dt = {} & \Pr(\text{failed at } t + dt \,|\, \text{good at } t) \times \Pr(\text{good at } t) \\ & + \Pr(\text{failed at } t + dt \,|\, \text{bad at } t) \times \Pr(\text{bad at } t)\end{aligned} \quad 14\text{-}1$$

The NMS Customer Survey gathered data on scrap rates amongst the population of firms using measurement services supplied by the NMS laboratories.[33] This survey found that the average scrap rate amongst these firms was 3.7%.

The "rebate rate" is the proportion of output that is returned by customers because it is defective. This will relate to defective output that wasn't caught during the conformance testing process and so got sold to customers. All such defective output will be returned by customers along with demands for a refund.[34] Hence, the "rebate rate" can be defined as follows:

$$(\text{rebate rate}).dt = \Pr(\text{passed at } t + dt \,|\, \text{bad at } t) \times \Pr(\text{bad at } t) \quad 14\text{-}2$$

Warranty Weekly compiled data on average claims rates amongst US manufacturers between 2003 and 2018. They found that 1.5% was the average percentage of sales revenue spent on warranty claims over this 16-year period.[35] It has no been possible to find specific estimates for the UK, but this US estimate provides a useful benchmark. Moreover, given that the UK and US have extensive (and roughly balanced) two-way trade flows, this US benchmark of 1.5% provides a reasonable estimate of the UK's claims rate.

Lastly, "success" should be understood as the production of viable goods that don't fall prey to type-1 errors in the testing process. Let us refer to the production of revenue generating goods as '*the production of viable output that generates revenue when it's sold.*' Hence, the "success rate" can be defined as follows:

$$(\text{success rate}).dt = \Pr(\text{passed at } t + dt \,|\, \text{good at } t) \times \Pr(\text{good at } t) \quad 14\text{-}3$$

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[33] See Table 27 (page 92) of the final report supplied by Winning Moves.
[34] The rebate rate will closely correspond to the proportion of revenue set-aside to cover warranty claims.
[35] https://www.warrantyweek.com/archive/ww20230316.html.

14.2 THREE COMPONENTS OF TOTAL OUTPUT

Although, $f[k(t)]$ is the total output from production, malfunctions in the production process and mistakes in conformance testing mean that not all this output will reach a buyer. Furthermore, even some of the output that does reach a buyer will be returned as defective. (The rejects being defective output that slipped through the conformance testing process due to type-2 errors.)

The three components of total output are: (1) viable output that is successfully sold to a customer; (2) output that is scrapped before it ever reaches a customer; and (3) defective output that is returned by unhappy customers wanting a rebate. (The terms ‘rebate’ and ‘refund’ are used interchangeably.)

Next, recall that $v(t) = \Pr(\text{good at } t)$ and $1 - v(t) = \Pr(\text{bad at } t)$. In other words, $v(t)$ is the proportion of machines that are functioning correctly; and $1 - v(t)$ is the proportion of machines that are malfunctioning. Using the definitions of θ, ϕ and $v(t)$, we get the following expressions for the components of total output:

The “scrap rate”, $\Gamma(t)$, is given by:

$$\Gamma(t) = \theta v(t) + \phi[1 - v(t)]\,. \qquad \text{14-4}$$

The “rebate rate”, $\Omega(t)$, is given by:

$$\Omega(t) = (1 - \phi)[1 - v(t)]\,. \qquad \text{14-5}$$

The “success rate”, $\Lambda(t)$, is given by:

$$\Lambda(t) = (1 - \theta)v(t)\,. \qquad \text{14-6}$$

It can be shown that these three components must satisfy the following basic identity:

$$(1 - \theta)v(t) + \{\theta v(t) + \phi[1 - v(t)]\} + (1 - \phi)[1 - v(t)] = 1$$

In words, this identity says:

$$\text{success rate} + \text{scrap rate} + \text{rebate rate} = 1$$

Since the three components must sum to unity, it follows that together they fully account for what happens to the output from production. Furthermore, if the scrap rate is 3.7% and the rebate rate is 1.5%, then by subtraction the success rate must be 94.8%.

14.3 REVENUE

The usable (non-defective) output from production is $v(t)f[k(t)]$, where $f[k(t)]$ is the total output and $v(t)$ is the proportion of this total output that is usable (non-defective).

Let $y(t)$ denote the revenue generated from the usable output that is sold to customers. Note that this output is a little less than the usable output from production because the existence of type-1 errors means that some of it is scrapped even though it is perfectly fine. The revenue generated by the real economy, $y(t)$, will be its total output, $f[k(t)]$, multiplied by its “success rate”:

$$y(t) = (1 - \theta)v(t)f[k(t)]\,, \qquad \text{14-7}$$

where $(1 - \theta)v(t)$ is the “success rate” and $f[k(t)]$ is the total output.

14.4 EFFECTIVE TFP

Most macroeconomic analysis assumes that production technology operate flawlessly, meaning that machinery never malfunctions in a way that produces defective outputs.

However, this is unrealistic, because the economy's effective TFP, B, really depends on three distinct factors: Firstly, the sophistication of the technology used for production, which sets the maximum TFP that is theoretically attainable in a perfect world with no mistakes. Secondly, some proportion of output will need to be scrapped because malfunctions in the production process have led to it being defective. Finally, some of the viable output will be unnecessarily scrapped due to type-1 errors in the testing process.

As the analysis in this report develops, it will become increasingly helpful to differentiate between the TFP as set by the limits of the production technology, A, and the effective TFP as realised in real economy, B. Note that uncorrected malfunctions in the production process mean that $B < A$.

The total output from the real economy is $y = (1 - \theta)vf(k)$, whilst the revenue generated from production is $f(k) = Ak^{\beta}$, where A is the economy's maximum possible TFP using its technology. Thus, the revenue generated from production can be written as $y = Bk^{\beta}$, where the "effective TFP" is given by:

$$B(t) = A(1 - \theta)v(t). \qquad \text{14-8}$$

Whilst the maximum possible TFP, A, is fixed parameter, the effective TFP, $B(t)$, will vary depending on the reliability of production, $v(t)$. Furthermore, $B(t)$ will also be influenced by the regret rate, θ, which itself depends on the likelihood of type-1 errors and the frequency of inspections.

14.5 WAGES AND RENTS

The marginal products of labour and capital are $\mathrm{MPL} = \partial Y/\partial L$ and $\mathrm{MPK} = \partial Y/\partial K$, respectively. Factor markets are competitive, and so labour and capital are paid their marginal products. Thus, the wage rate is $w = \partial Y/\partial L$, and the rental rate is $r = \partial Y/\partial K$. These partial derivatives can be evaluated to give the following results:

$$w = \partial Y/\partial L = \alpha(1 - \theta)vf(k), \qquad \text{14-9}$$

$$r = \partial Y/\partial K = (1 - \theta)vf_k(k), \qquad \text{14-10}$$

where

$$f_k(k) = df/dk = \beta f(k)/k.$$

Furthermore, if k is the capital per worker (capital intensity) and $y = (1 - \theta)vf(k)$ is the output per worker (labour productivity), then the wage income of the representative citizen can be written as $w = \alpha y$, and their rental income can be written as $rk = \beta y$.

Proof. If y is the per capita output and L is the size of the workforce, then the economy's aggregate output (GVA) must be: $Y = yL$. So, starting from this identity, the result for the wage rate can be found as follows. Since $w = \partial Y/\partial L$, where $Y = yL$, the chain-rule of differentiation gives:

$$w = \frac{\partial(Ly)}{\partial L} = y + L\frac{\partial y}{\partial L}, \text{where } \frac{\partial y}{\partial L} = (1 - \theta)vf_k(k)\frac{\partial k}{\partial L}.$$

Since $k = K/L$, it follows that: $\partial k/\partial L = -K/L^2 = -k/L$. From which, the expression for $\partial y/\partial L$ becomes:

$$\frac{\partial y}{\partial L} = -\frac{1}{L} \times (1 - \theta)vf_k(k)k.$$

And, since $f_k(k)k = \beta f(k)$, we arrive at: $\partial y/\partial L = -\beta y/L$. Substituting this result back into our earlier expression for w yields:

$$w = y - L(\beta y/L) = (1 - \beta)y.$$

Lastly, using constant returns-to-scale, this gives the main result: $w = \alpha y$.

Since $r = \partial Y/\partial K$, where $Y = yL$, the result for the rental rate is almost immediate:

$$r = L\frac{\partial y}{\partial K}, \text{where } \frac{\partial y}{\partial K} = (1-\theta)vf_k(k)\frac{\partial k}{\partial K}.$$

Since $k = K/L$, the expression for $\partial y/\partial K$ becomes:

$$\frac{\partial y}{\partial K} = \frac{1}{L} \times (1-\theta)vf_k(k).$$

Substituting this back into the expression for r yields the main result: $r = (1-\theta)vf_k(k)$. Multiplying the rental rate, r, by capital per capita, k, gives the representative citizen's rental income: $rk = (1-\theta)vf_k(k)k$. Lastly, since $f_k(k)k = \beta f(k)$, this becomes: $rk = \beta y$. ■

It has been shown that constant returns-to-scale and competitive factor markets determine how the output from production is split between labour and capital: If $y(t)$ is the per capita output and $k(t)$ is the economy's capital intensity, then a worker's wage is $w(t) = \alpha y(t)$, and a capital owner's rental income is $r(t)k(t) = \beta y(t)$, where α and β are the indexes of labour and capital from the Cobb-Douglas production function. Notice that due to constant returns-to-scale, adding both these sources of income together yields the per capita output of the economy (as it must).

At this point, we have most of the quantities needed to set up the model. The next few sections derive differential equations for the evolution of the capital stock and the reliability of production.

15 EVOLUTION OF THE CAPITAL STOCK

As already discussed, the output that's available for either consumption or investment will be less than the usable output for two reasons: Firstly, the revenue generated by selling goods is $y(t)$. This is less than the total output from production, because of malfunctions in the production process and because some of the viable output falls prey to type-1 errors. Secondly, the output used to pay for the services of CT engineers isn't available for consumption or investment.

This section gives an equation for the evolution of the capital stock, much like that found in the Solow model, but that now accounts for the two influences mentioned above.

15.1 THE NET REVENUE

As described earlier, there is assumed to be an international market for engineers, where ω is the wage that an engineer commands in this market. Furthermore, engineers are highly mobile "citizens of the world", so that engineering services can be imported from outside of the economy if demand exceeds supply.

The per capita cost of buying the engineers' testing services is $\omega e(t)$, where $e(t)$ is the number of engineers per capita and ω is their wage rate. As discussed, τ denotes the average cost of employing CT engineers to supervise one million pounds of capital equipment. And, from the definition of $k(t)$ (capital intensity), it follows that:

$$\tau = [\omega e(t)]/k(t). \qquad \text{15-1}$$

Hence, the cost of employing these CT engineers can be written as follows:

$$\text{income earned by CT engineers} = \omega e(t) = \tau k(t). \qquad \text{15-2}$$

The cost of paying engineers to supervise production uses up some of the output that would otherwise be available for either consumption or investment. Hence, the "net-revenue" from production is the revenue minus the wages of the CT engineers:

$$\text{net revenue} = \text{revenue} - \text{income earned by CT engineers}$$

Let $y_\dagger(t)$ denote the net revenue, so that the previous equation can be rewritten as:

$$y_\dagger(t) = y(t) - \tau k(t)\,. \qquad \text{15-3}$$

Since $y(t) = (1 - \theta)v(t)f[k(t)]$, the per capita output that's available for consumption or investment (net-revenue) becomes:

$$y_\dagger(t) = (1 - \theta)v(t)f[k(t)] - \tau k(t), \qquad \text{15-4}$$

where the total output per worker is $f(k) = Ak^\beta$.

15.2 A FLAT TAX ON INCOMES TO PAY FOR CT ENGINEERS

The CT engineers don't directly generate products that can be sold, and so their activities take place outside of the real economy. (The output of their efforts - valuable as it is - cannot be used to feed, clothe, or shelter the population.) Rather, the CT engineers create a sense of security that helps the efficient functioning of the real economy. However, the CT engineers won't work for free, and so they require a small portion of the revenue from production as payment.

The real economy serves society, and society needs a mechanism to pay for the work of the CT engineers. Ultimately, citizens must pay for the benefits of having reliable production processes and the security of knowing that the goods they buy aren't defective. There are many institutional arrangements through which the payments could be collected.

- A mandatory MOT for capital equipment.
- A regulation stipulating that for every million pounds of capital equipment, a firm must employ a certain number of CT engineers.
- A tax on incomes that pays for CT engineers to carry out inspections. (Rather like paying for Public Analysts through taxation.)

Of these, the simplest to model is a flat tax on citizens' incomes deducted at source by firms and paid to the engineers. (One could think of it as being a bit like the collection of tax through the PAYE system.) Moreover, the workers are also the owners of the economy's capital, and so the split between "taxes" on labour and "taxes" on capital isn't critical to the model. At an aggregate level, the complexity of the various institutional arrangements will washout to result in the same claim on society's resources.

Let $m(t)$ denote the money collected through a "reliability tax" as a proportion of the total output from production. However, these payments are really made voluntarily by self-interested producers wanting to avoid accidentally supplying defective outputs to their customers. From a macroeconomic perspective, it resembles a kind of yearly "sacrifice" made by producers to keep the negative effects of entropy at bay.

As discussed, τ is the cost of paying CT engineers to supervise one million pounds of capital for a year. If the output of the real economy is $Y(t)$ and the capital stock is $K(t)$, then the tax ratio needed to pay for the services of the CT engineers is $m(t) = \tau K(t)/Y(t)$. This shows that, for a fixed value of τ, the required tax ratio is proportional to the economy's capital ratio. Expressed using the intensive forms of $Y(t)$ and $K(t)$, gives us an expression for the tax ratio with respect to this "reliability tax":

$$m(t) = \tau k(t)/y(t)\,. \qquad \text{15-5}$$

Notice that $m(t)$ corresponds to the proportion of the economy's GVA that is spent on paying the wages of CT engineers. That is, $m(t)$ is the cost of conformance testing as a proportion of the economy's total output.

Some of the output from production is necessarily used up through employing CT engineers to supervise production. Furthermore, the cost of supervising production must be split between the two factors of production: Applying this tax rate to someone's income yields a levy of $\alpha\tau k(t)$ from their wage income and a levy of $\beta\tau k(t)$ from their rental income. That is, for the supervision of a unit of capital worth one million pounds, labour pays $\alpha\tau$ and capital pays $\beta\tau$, so that the cost is split between labour and capital in the same proportion as the revenue. (In our model, the split between labour and capital is immaterial because citizens are both "workers" and "capitalists".)

The "net wage" and "net rent" received by our representative citizen can be found by subtracting these levies from their wage and rental income. Their net wage is given by:

$$w_\dagger(t) = w(t) - \alpha\tau k(t) \,. \qquad \text{15-6}$$

Their net rental income is $r_\dagger(t)k(t) = r(t)k(t) - \beta\tau k(t)$, which implies that the "net rental rate" is given by:

$$r_\dagger(t) = r(t) - \beta\tau \,. \qquad \text{15-7}$$

In other words, $r_\dagger(t)$ is the after-tax return on a unit of capital.

Substituting for $w(t)$ and $r(t)$ (using the formulae above) gives the following expressions for the net wage rate and the net rental rate:

$$w_\dagger(t) = \alpha[(1-\theta)v(t)f[k(t)] - \tau k(t)] \,,$$

$$r_\dagger(t) = (1-\theta)v(t)f_k[k(t)] - \beta\tau \,.$$

Notice that because $\beta f[k(t)] = f_k[k(t)]k(t)$, the wage rate and rental rate are connected as follows:

$$w_\dagger(t) = \frac{\alpha}{\beta} r_\dagger(t)k(t) \qquad \text{15-8}$$

Finally, the aggregate revenue from production must be divided between labour (wages), capital (rents), and payments to the CT engineers:

$$\text{aggregate revenue} = \text{net wages} + \text{net rent} + \text{cost of CT engineers}$$

Hence, the per capita output, $y(t)$, is divided between someone's wage, the rents they receive as the owners of capital, and "taxes" paid for the services of CT engineers. The intensive form of this decomposition of someone's income is as follows:

$$y(t) = w_\dagger(t) + r_\dagger(t)k(t) + \tau k(t) \,, \qquad \text{15-9}$$

where $w_\dagger(t)$ is the net wage rate, $r_\dagger(t)$ is the net rental rate, and $\tau k(t)$ is the cost of employing the CT engineers (paid for through a flat tax on citizens' incomes).

15.3 AN EQUATION FOR THE EVOLUTION OF THE CAPITAL STOCK

The net-revenue from production can be used for consumption or investment. In the Solow model, a fixed fraction of output, s, is saved so that it can be invested in the capital stock.

In the Solow model, "output" and "net-revenue" are one and the same thing, but in our model the use of an imperfect production process, along with payments to CT engineers, means that net-revenue is less than the output.

Swapping output for net-revenue gives us a modified version of Solow's equation for the evolution of the per capita capital stock:

$$\dot{k}(t) = s[y(t) - \tau k(t)] - (\delta + g)k(t), \qquad \text{15-10}$$

where s is the exogenously set savings rate and $y(t)$ is the revenue earned from selling viable goods. Note that the condition for $\dot{k}(t) = 0$ says that: '*in equilibrium, gross savings equals gross investment*.' This is, recognisably, a version of the state equation from the Solow model, but which now incorporates the loss due to imperfect production processes, as well as the costs incurred due to conformance testing.

15.4 GROSS INVESTMENT AND NET INVESTMENT

The "net investment rate" equals the yearly growth in the capital stock as a proportion of the existing capital stock, which can be written as $\dot{K}(t)/K(t)$, where $\dot{K}(t) \equiv dK/dt$ and $K(t)$ is the value of the existing capital stock. Furthermore, given that $k(t) = K(t)/L(t)$ and $\dot{L}(t)/L(t) = g$, we arrive at the following identity for the "net investment rate":

$$\frac{\dot{K}(t)}{K(t)} = \frac{\dot{k}(t)}{k(t)} + g \qquad \text{15-11}$$

The flow of "gross investment" equates to the yearly increase in the capital stock, $\dot{K}(t)$, plus any further investment that's used to offset the effects of depreciation. Thus, the flow of gross investment is $\dot{K}(t) + \delta K(t)$, and so the "gross investment rate" becomes $\dot{K}(t)/K(t) + \delta$. Hence, we arrive at the following identity for the "gross investment rate":

$$\frac{\dot{K}(t)}{K(t)} + \delta = \frac{\dot{k}(t)}{k(t)} + g + \delta. \qquad \text{15-12}$$

15.5 THE GROWTH RATE OF THE ECONOMY'S CAPITAL INTENSITY

The equation for $\dot{k}(t)$ can be rearranged to give an expression for the growth rate of the economy's capital intensity. Since $y(t) = (1 - \theta)v(t)f[k(t)]$ and $\beta f[k(t)] = f_k[k(t)]k(t)$, it follows that:

$$y(t) = \frac{1}{\beta}(1 - \theta)v(t)f_k[k(t)]k(t)$$

Since $r(t) = (1 - \theta)v(t)f_k[k(t)]$, this can be rewritten in terms of the rental rate:

$$y(t) = \frac{1}{\beta}r(t)k(t) \qquad \text{15-13}$$

So, upon substituting for $y(t)$ in our equation for $\dot{k}(t)$ gives:

$$\dot{k}(t) = \frac{s}{\beta}[r(t) - \beta\tau]k(t) - (\delta + g)k(t).$$

From which we get the following expression for the growth rate of the economy's capital intensity:

$$\frac{\dot{k}(t)}{k(t)} = \frac{s}{\beta}[r(t) - \beta\tau] - (\delta + g). \qquad \text{15-14}$$

The intuition behind this formula starts to become more apparent when it's rewritten as:

$$\frac{\dot{k}(t)}{k(t)} + g = \frac{s}{\beta}[r(t) - \beta\tau] - \delta.$$

Firstly, recall that $\dot{K}(t)/K(t) = \dot{k}(t)/k(t) + g$, and so the LHS of the equation is the net change in the capital stock as a percentage of the existing capital stock. Secondly, consider the RHS of the equation, where the expression in square brackets is recognisable as the net rental rate, $r_{\dagger}(t) = r(t) - \beta\tau$.

- Dividing the net rental rate, $r_{\dagger}(t)$, by capital's share of income, β, gives the usable output generated for each unit of capital employed.
- Multiplying the usable output by the economy's savings rate, s, gives the gross investment in new capital for each unit of capital in the existing capital stock.

Together these bullet points imply that the first term on the RHS gives the gross investment rate.

Lastly, the negative term on the RHS represents that fraction of existing capital that is lost due to depreciation, meaning that the expression on the RHS gives the net investment rate.

16 THE EVOLUTION OF RELIABILITY

As discussed, $v(t)$ is the proportion of machines that are functioning correctly at time t; and $1 - v(t)$ is the proportion of machines that are malfunctioning at time t. Let $\dot{v}(t)$ denote the derivative of $v(t)$ with respect to time, t. In other words, $\frac{dv}{dt} = \dot{v}(t)$ is the change in $v(t)$ that occurs during the interval of time between t and t + dt.

A machine is either in a "good" state (so that it's functioning correctly) or in a "bad" state (so that it's malfunctioning). There is a positive influence on $v(t)$ from the inflow of "good" machines; and there is a negative influence on $v(t)$ from the outflow of previously "good" machines that have gone "bad". The change in $v(t)$, at time t, is determined by the net-effect of these two opposing influences:

$$\frac{dv}{dt} = \dot{v}(t) = \text{inflow of good machines at } t - \text{outflow of good machines at } t$$

The outflow corresponds to a small portion of the "good" machines going "bad". As already discussed, this outflow is $\varepsilon v(t)$, where ε is the transition rate. However, more effort is needed to find an expression for the inflow.

There are two types of positive influence, each contributing to the inflow of "good" machines:

- The investment in new capital equipment, where capital items are assumed to function perfectly when first installed. (The gross investment rate is the net-investment rate plus the investment needed to offset the effects of depreciation.)
- Engineers search for malfunctioning machines, and then reset them, so that these machines once again work correctly. As discussed already, the likelihood of a malfunctioning machine being reset corresponds to the "detection rate", ϕ.

The aim of this section is to derive an expression for the inflow of "good" machines. This section proceeds by considering two special cases and then combining the results to construct an expression for the general case.

16.1 THE SPECIAL CASE WITHOUT CONFORMANCE TESTING

In this sub-section let us assume that machines are not being reset by CT engineers. That is, this sub-section considers the special case where $E(t) = 0$.

Since $K(t)$ denotes the quantity of machines in the capital stock and $v(t)$ denotes the proportion of these machines that are still working correctly, it follows that the quantity of "good" machines is vK. The change in the quantity of "good" machines can be found by differentiating vK with respect to time, t, and the chain-rule of differentiation yields the following expression:

$$\frac{d}{dt}(vK) = \dot{v}K + v\dot{K}$$

This identity can be turned into a useful equation through a consideration of three distinct influences on the stock of "good" machines:

(1) The in-flow of "good" machines because of gross investment.
(2) The out-flow of "good" machines because of depreciation.
(3) The out-flow of "good" machines because of the transition rate.

Firstly, let us suppose that all new machines are born "good", in the sense that following installation they always work correctly after the original set-up. This implies that the in-flow of "good" machines is the gross investment: $\dot{K} + \delta K$, where $\dot{K}$ is the net-investment, and δK is what is needed to offset the effects of depreciation. Secondly, some of the "good" machine expire due to depreciation. If we regard depreciation as attrition from random breakages, then the broken machines will be representative of the capital stock. In particular, the proportion of "good" machines amongst those that break will be the same as the proportion of "good" machines amongst the whole of the capital stock. This implies that an expression for the "good" machines lost to depreciation each year is $\delta v K$. Lastly, because of the transition rate, a small proportion, ε, of the "good" machines will go "bad" each year.

The change in the number of good machines during a year will be the in-flow of new machines minus the outflow of previously "good" machines that broke or went "bad" during the year. Combining these insights with the identity above gives us the following equation:

$$\dot{v}K + v\dot{K} = \dot{K} + \delta K - \delta v K - \varepsilon v K$$

This equation can be rearranged to give us an expression for the change in v with respect to time:

$$\dot{v} = \left(\delta + \frac{\dot{K}}{K}\right)(1 - v) - \varepsilon v$$

Lastly, as $k = K/L$ and $\dot{L}/L = g$, we get the following identity: $\dot{k}/k = \dot{K}/K - g$. So, after substituting for $\dot{K}/K$, the equation above becomes:

$$\dot{v} = \left(\delta + g + \frac{\dot{k}}{k}\right)(1 - v) - \varepsilon v \qquad \text{16-1}$$

The second term implies that adding new machines raises v (because it's always the case that $v < 1$). In the unrealistic situation where $v = 1$, all existing machines are "good", and so this term vanishes, as you can't improve on perfection. The final term is necessarily negative since it represents the continual decay of the capital stock, because with each passing year a proportion of machines either break or go from "good" to "bad". Hence, this equation shows how the decay of $v(t)$ can be offset by the in-flow of new machines through the gross investment.

16.2 THE SPECIAL CASE WITHOUT INVESTMENT

In this subsection, let us simplify the situation by imagining, for a while, that there is no longer a positive in-flow of new machines due to investment. That is, this subsection considers the special case where gross investment is zero: $\dot{K}(t) + \delta K(t) = 0$.

Suppose that at time t there are $E(t)$ engineers supervising production and that some of the machines are already malfunctioning. Let $v(t)$ denote the proportion that are working correctly, so that the proportion that are malfunctioning is $1 - v(t)$. In the special case under consideration, the change in $v(t)$ at time *t* depends on both the probability of "bad" machines being fixed and of "good" machines going "bad".

The engineers search through the machines in the capital stock looking for those that are malfunctioning, and so need to be reset. Before conducting conformance tests on a machine, an engineer doesn't know whether a given machine is malfunctioning or not. Hence, the CT engineers will encounter these malfunctioning machines at random as they search through the capital stock. Thus, the likelihood of an engineer encountering a malfunctioning machine is $1 - v(t)$.

Bayes Theorem implies that the change in $v(t)$ that occurs between time t and t + dt is given by:

$$\dot{v}(t).dt = \Pr(\text{failed } t + dt | \text{ bad at } t)[1 - v(t)] - \Pr(\text{bad at } t + dt | \text{ good at } t)v(t),$$

On the RHS of this expression, the first term represents the in-flow from the malfunctioning machines that are found, and fixed, by CT engineers because of their conformance testing endeavours. The second term represents the out-flow of previously "good" machines when they start malfunctioning.

As already discussed, the transition rate is $\varepsilon.dt = \Pr(\text{bad at } t + dt | \text{ good at } t)$. So, using this definition of the transition rate, the previous expression becomes:

$$\dot{v}(t).dt = \Pr(\text{failed } t + dt | \text{ bad at } t)[1 - v(t)] - \varepsilon v(t).dt$$

Furthermore, the detection rate is $\phi.dt = \Pr(\text{failed } t + dt | \text{ bad at } t)$. So, in the special case where there is no investment, the change in $v(t)$ becomes:

$$\dot{v}(t) = \phi[1 - v(t)] - \varepsilon v(t) \qquad \text{16-2}$$

Hence, the change in $v(t)$ depends on both the current level of $v(t)$ and the degree to which the capital stock is being supervised by CT engineers.

Lastly, it's important to keep in mind that this equation is only valid for the special case in which there's no gross investment. (In the special case where something is stopping any new machines from being added into the capital stock.)[36]

16.3 THREE INFUENCES ON THE RELIABILITY OF PRODUCTION

The final step in the derivation is to combine the formulae derived in the two previous sub-sections to get a single formula that accounts for all three of the processes governing changes in $v(t)$:

- The decay of $v(t)$ due to instability in the production process.
- The in-flow of new machines due to gross investment.
- The resetting of "bad" machines by the engineers.

Let us take each bullet point in turn and summarise the relevant results from our previous analysis.

Instability in Production: The first bullet point concerns malfunctions in the production process that cause previously "good" machines to go "bad". It has already been explained that the decay of $v(t)$ due to malfunctioning machines is given by $-\varepsilon v(t)$.

In-Flow of New Machines: The second bullet point concerns the in-flow of "good" machines due to investment, where the argument runs as follows: As already discussed, the gross investment rate can be written as: $\delta + g + \dot{k}(t)/k(t)$. The positive effect of installing new machines can then be found by multiplying this gross investment rate by the proportion of machines that have gone "bad". That is, the contribution from the in-flow of new machines is given by $[\delta + g + \dot{k}(t)/k(t)][1 - v(t)]$, where $\delta + g + \dot{k}(t)/k(t)$ is an expression for the

[36] The formula in this sub-section is only valid in the special case where $\dot{K}(t) + \delta K(t) = 0$, which implies that $\dot{K}(t)/K(t) + \delta = 0$. Notice that, as $\dot{k}(t)/k(t) = \dot{K}(t)/K(t) - g$, this equates to: $\dot{k}(t)/k(t) + \delta + g = 0$.

gross investment rate and $1 - v(t)$ is the proportion of machines that are currently malfunctioning.

Resetting "Bad" Machines: The final bullet point concerns the use of conformance testing to find, and then reset the "bad" machines. In the second subsection, it was shown that the effect of these resets is captured by multiplying the detection rate by the proportion of machines that have gone "bad". That is, the positive influence of resets is $\phi[1 - v(t)]$, where ϕ is the detection rate and $1 - v(t)$ is the proportion of machines that are malfunctioning.

Given that instabilities, investments, and resets are all separate influences on $v(t)$, the expressions given above can be added together to get a formula that accounts for all these influences. The general formula for $\dot{v}(t)$ is as follows:

$$\dot{v}(t) = \text{inflow from investment} + \text{inflow from resets} - \text{outflow from malfunctions}$$

Where, the formulae for these components are:

$$\text{inflow from investment} = \left[\dot{k}(t)/k(t) + \delta + g\right][1 - v(t)] \qquad \text{16-3}$$

$$\text{inflow from resets} = \phi[1 - v(t)] \qquad \text{16-4}$$

$$\text{outflow from malfunctions} = \varepsilon v(t) \qquad \text{16-5}$$

Using the findings from each of the special cases (as listed above) to substitute for the elements in this formula gives us the following result:

$$\dot{v}(t) = \left[\frac{\dot{k}(t)}{k(t)} + \delta + g + \phi\right][1 - v(t)] - \varepsilon v(t) \qquad \text{16-6}$$

This is the general equation for the evolution of $v(t)$ and it is valid in all situations.

17 THE STEADY STATE OF THE SYSTEM

This section sets up and solves an extended version of the Solow model for an economy with imperfect production processes and conformance testing. For now, this involves making the inspection frequency, n, a constant parameter whose value is determined by factors beyond the model. (Later on in this report, n will be endogenized by deriving an optimality condition for the amount that businesses spend on conformance testing.)

This adapted version of the Solow model leads to a system of two differential equations: one for the evolution of the reliability of the production process, $v(t)$, and another for the evolution of the capital intensity, $k(t)$. The system is deterministic in the sense that together the parameters and initial conditions determine which trajectory is followed by the economy.

17.1 A SYSTEM OF COUPLED DIFFERENTIAL EQUATIONS

This subsection summarises the model. Let us begin by reminding ourselves what the parameters θ, ϕ, and τ represent:

- θ is the proportion of usable output (non-defective goods) that is nonetheless scrapped because it falls prey to type-1 errors in the testing process. Hence, $1 - \theta$ is the proportion of usable output that makes it through the conformance testing process.
- ϕ is the proportion of "bad" machines in the capital stock that are found by engineers and then reset so that they, once again, work correctly.
- τ is the supervision cost for one million pounds of capital. That is, it's the cost of employing engineers to supervise a unit of capital, where capital is measured in millions of pounds.

In per capita terms, the main elements of the model can be summarised as follows: The revenue from production is given by $y(t) = (1 - \theta)v(t)f[k(t)]$, where $f(k) = Ak^{\beta}$ is the intensive form of the production function. However, due to the cost of paying engineers to supervise capital equipment, the net revenue from production becomes $y_{\dagger}(t) = y(t) - \tau k(t)$, where $\tau k(t)$ is the "reliability tax" paid by the representative citizen.

Constant returns-to-scale and competitive factor markets imply that the rental rate equals the marginal product of capital (MPK), and so $r(t) = (1 - \theta)v(t)f_k[k(t)]$, where $f_k(k) = df/dk$. However, due to the cost of employing CT engineers, the net rental rate is $r_{\dagger}(t) = r(t) - \beta\tau$, where $\beta\tau$ represents the "reliability toll" as applied to capital.

Some important formulae - identities that feature in the model - are as follows:

- The "reliability tax", as a proportion of total output, is given by $m(t) = \tau k(t)/y(t)$; which is equivalent to the proportion of revenue spent on conformance testing.
- Since citizens save part of their income, consumption becomes $c(t) = (1 - s)y_{\dagger}(t)$, where s is the saving rate and $y_{\dagger}(t)$ is "post-tax" income.
- The economy's capital ratio becomes $k(t)/y(t) = \beta/r(t)$, where $r(t)$ is the rental rate and β is the index for capital from the Cobb Douglas production function.

The dynamics of the economy are determined by the following pair of differential equations:

$$\dot{v}(t) = \left[\frac{\dot{k}(t)}{k(t)} + \delta + g + \phi\right][1 - v(t)] - \varepsilon v(t), \qquad 17\text{-}1$$

$$\frac{\dot{k}(t)}{k(t)} = \frac{s}{\beta}r(t) - (\delta + g + s\tau), \qquad 17\text{-}2$$

where $r(t)$ is the rental rate. The first equation is for the reliability of production processes, and the second equation governs the rate of capital accumulation.

17.2 THE STEADY STATE OF THE SYSTEM

The dynamics of the system can be represented in a two-dimensional phase space, whose axes are the economy's capital intensity and the reliability of production. Using this concept of a phase space, the dynamics of the system can be analysed to characterise the nature of any equilibria.

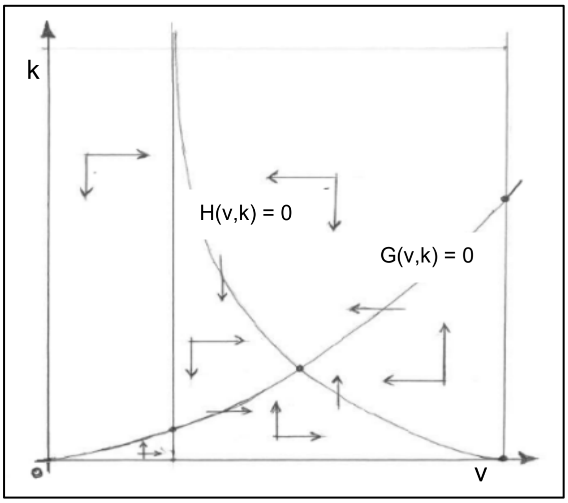

Figure 2: Split the phase space into regions using G(v,k) = 0 and H(v,k) = 0.

In the figure above, the dynamics of the system have been pictured in a two-dimensional phase diagram, in which the axes are the economy's capital intensity (vertical axis) and the reliability of production (horizontal axis).

The dynamics can be represented through a system of first-order differential equations:

$$\dot{v} = H(v, k),$$

$$\dot{k} = G(v, k).$$

Where details of $H(\cdot)$ and $G(\cdot)$ are given in Annex B. The Jacobian of this system can be written as:

$$\mathfrak{J}(v, k) = \begin{pmatrix} \frac{\partial H}{\partial v} & \frac{\partial H}{\partial k} \\ \frac{\partial G}{\partial v} & \frac{\partial G}{\partial k} \end{pmatrix}$$

Qualitative analysis requires an understanding of the orbits of the system around its fixed point $(\boldsymbol{v}^*, \boldsymbol{k}^*)$. According to the Hartman-Grobman (HG) Theorem, the orbits of a dynamical system, in the neighbourhood of a fixed point, are equivalent to those of the linearised system providing that no eigenvalue of the linearised system has real part equal to zero. It can be shown that this condition holds for our system providing that neither the trace nor the determinant of the Jacobian are zero. It can be further shown that:

1. The Conditions for the HG theorem will always be satisfied by our system.
2. The fixed point will always be a sink and thus represents the steady state of the system
3. The paths converging towards the steady state can be either monotonic or spiral depending on the specifics of the parameter values.

This phase space analysis (detailed in Annex B) shows that there is a unique stable equilibrium on which all paths ultimately converge on the fixed point.

Proposition 17-1: The system is in equilibrium when $\boldsymbol{v(t)}$ and $\boldsymbol{k(t)}$ are such that $\boldsymbol{\dot{v}(t)} = \boldsymbol{0}$ and $\boldsymbol{\dot{k}(t)} = \boldsymbol{0}$. Firstly, it can be shown that the system has a unique equilibrium (steady state), determined by the fixed parameters of the economy. Secondly, the economy will end up settling into this equilibrium regardless of where it begins. That is, all trajectories converge towards this unique and stable equilibrium.

Let v^* and k^* denote the equilibrium values of $v(t)$ and $k(t)$, respectively. The equilibrium values can be substituted into formulae for $r(t)$ and $y(t)$ to find the equilibrium values of the rental rate and the revenue from production; according to which the steady state value of the rental rate and revenue becomes:

$$\boldsymbol{r}^* = (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{v}^*\boldsymbol{f}_{\boldsymbol{k}}(\boldsymbol{k}^*), \quad \text{17-3}$$

$$\boldsymbol{y}^* = (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{v}^*\boldsymbol{f}(\boldsymbol{k}^*). \quad \text{17-4}$$

In the steady state, all quantities must remain finite and positive. For example, there cannot be an equilibrium in which $k(t) = 0$, given that $f_k(k) \to \infty$ as $k \to 0$. Note that $f_k(k) \to \infty$ implies an infinite rental rate. More generally, for there to be a functioning economy that produces some output, it must be the case that $v(t) > 0$ and $k(t) > 0$. In particular, these positivity assumptions must be satisfied in equilibrium, which implies that $v^* > 0$ and $k^* > 0$.

The conditions for this unique stable equilibrium are as follows:

$$\dot{v}(t) = 0 \text{ and } \dot{k}(t) = 0$$

if and only if

$$(\delta + g + \phi)[1 - v(t)] - \varepsilon v(t) = 0, \quad \text{17-5}$$

$$\frac{s}{\beta} r(t) - (\delta + g + s\tau) = 0. \quad \text{17-6}$$

The next step is to use these equilibrium conditions to solve for the equilibrium values of the quantities in the model. And, because there's a unique steady state, it follows that:

$\dot{v}(t) = 0$, $k(t) = 0$ if and only if $v(t) = v^*$, $k(t) = k^*$.

Hence, these equilibrium conditions can be expressed in terms of the equilibrium values of the quantities involved:

$$(\delta + g + \phi)(1 - v^*) = \varepsilon v^*, \quad \text{17-7}$$

$$r^* = \frac{\beta}{s}(\delta + g + s\tau). \quad \text{17-8}$$

The first equation is a necessary and sufficient condition for $\dot{v}(t) = 0$; and the second equation is a necessary and sufficient condition for $\dot{k}(t) = 0$.

To summarise, the condition for $\dot{v}(t) = 0$ says that: '*in equilibrium, the inflow of "bad" machines equals outflow of "bad" machines.*' Next, the condition for $\dot{k}(t) = 0$ says that: '*in equilibrium, the income from rents equals capital's share of the output from production*.' This interpretation for the equation for r^* follows from further analysis showing that, in equilibrium, the capital intensity is $k^*/y^* = s/(\delta + g + s\tau)$, and so by multiplying the expression for r^* by k^* we get an expression for the income from rents.

17.3 THE RELIABILITY OF PRODUCTION PROCESSES

Imagine that there are two "buckets": one containing well-functioning machines (the "good" bucket) and the other containing the malfunctioning machines (the "bad" bucket). The figure below depicts the flows of probability into and out of these buckets when the system is in equilibrium. The inwards arrows denote positive contributions and the outwards arrows the leakages.

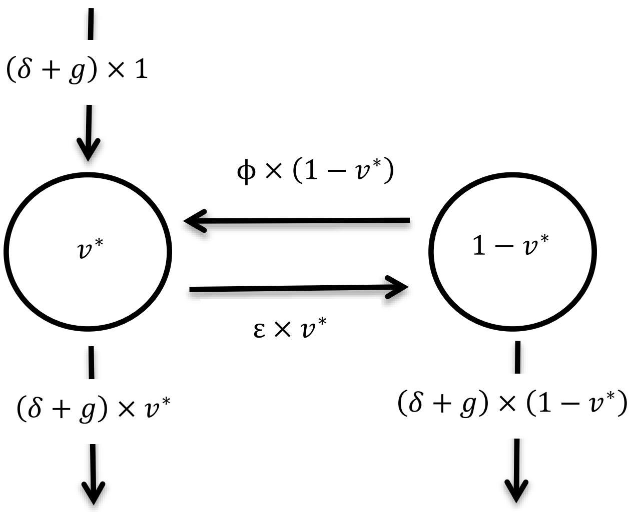

Figure 3: Flows of probability in and out of v^* and $1 - v^*$

Adding the inflows and subtracting the outflows gives the net flow. The net flow into the "good" bucket is given by $\phi(1-v^*)+(\delta+g)\ -(\delta+g)v^*-\varepsilon v^*$, which then simplifies to $(\delta+g+\phi)(1-v^*)-\varepsilon v^*$. Moreover, since the system is in equilibrium, the inflow must equal the outflow, which requires $(\delta+g+\phi)(1-v^*)=\varepsilon v^*$. So, the exact same relation holds for the "bad" bucket.

Thus, for the system to be in equilibrium, there must also be equal flows in and out of the "bad" bucket; and, if one bucket is held at a steady level, then so is the other bucket. Hence, we can interpret the diagram above as representing the steady state of a Markov process.

To better explore the economics behind our graphical analysis, we rewrite the equilibrium condition for $\dot{v}(t)=0$ as follows:

$$\varepsilon v^*=(\delta+g)(1-v^*)+\phi(1-v^*), \qquad 17\text{-}9$$

where $\delta+g$ is the "replacement rate" for capital items in the system's steady state. Each element in this equation has an economically meaningful interpretation:

- εv^* represents the "good" machines that go "bad".
- $(\delta+g)(1-v^*)$ represents the "bad" machines that exist in the capital stock and are replaced by new machines that enter the capital stock. Where, this "replacement" happens through the steady state investment required to offset the effects of depreciation, as well as to meet the need for extra tools due to growth in the workforce.
- $\phi(1-v^*)$ represents the "bad" machines that are found, and then fixed, by engineers as they work their way through the capital stock.

Hence, the LHS of the equation represents the inflow of "bad" machines, whereas the RHS of the equation represents the outflow of "bad" machines. Moreover, in an equilibrium, the inflow and the outflow must balance each other out, which is the underlying basis of this equilibrium condition.

Next, with a little rearrangement, our equilibrium condition for $\dot{v}(t)=0$ yields an expression for $1-v^*$:

$$1-v^*=\frac{\varepsilon}{\delta+g+\phi+\varepsilon} \qquad 17\text{-}10$$

Solving this equation for v^* gives the following expression:

$$v^*=\frac{\delta+g+\phi}{\delta+g+\phi+\varepsilon} \qquad 17\text{-}11$$

Which implies that the odds of a machine functioning correctly (of it being in a "good" state) become:

$$\frac{v^*}{1-v^*}=\frac{1}{\varepsilon}(\delta+g+\phi) \qquad 17\text{-}12$$

As expected, these odds increase if the engineers get better at finding the malfunctioning machines (an increase in ϕ); and these odds decrease if the transition rate were to increase (an increase in ε). Also, notice that the odds do not directly depend on either θ or τ.

Moreover, notice that v^* is independent of θ because it doesn't feature in equation 17-11, which implies that: $\partial v^*/\partial\theta=0$.

Next, recall that θ is a function of $\wp_{1|0}$, and ϕ is a function of $\wp_{1|1}$. So, an important implication of this result for v^* is that, in equilibrium, the reliability of production does not depend on the likelihood of type-1 errors, whereas it does depend on the likelihood of type-2 errors.

Lastly, since the reliability of the production process, $v(t)$, goes to v^* in the system's equilibrium, it follows that the effective TFP is given by:

$$B^* = A(1 - \theta)v^*$$

This shows that, in equilibrium, the effective TFP depends on the regret rate, θ, and the reliability of production processes, v^*.

17.4 A VERSION OF PIKETTY'S FORMULA

It has already been shown that the condition for $\dot{k}(t) = 0$, yields an expression for the equilibrium value of the rental rate:

$$r^* = \frac{\beta}{s}(\delta + g + s\tau) \qquad \text{17-13}$$

This is a modified version of Piketty's fundamental equation connecting capital's share of income, β, to the economy's savings rate, s. Note that there's an extra term within the bracketed expression that represents the cost of paying engineers to supervise production.

As discussed, the net rental rate becomes $r_\dagger^* = r^* - \beta\tau$, where $\beta\tau$ represents the "reliability tax" applied to capital. Hence, we get the following results for the equilibrium value of the net rental rate:

$$r_\dagger^* = \frac{\beta}{s}(\delta + g). \qquad \text{17-14}$$

On the LHS of this equation is the net rental rate, $r_\dagger^*$. This the annual income from renting out one unit of capital, r^*, less capital's contribution towards paying for engineers to supervise this unit of capital, $\beta\tau$. On the RHS is the bracketed term containing $\delta + g$. This represents the gross investment rate needed to keep pace with a growing workforce and the effects of depreciation. (Which is the "replacement rate" for capital in the steady state of the system.)

Notice that the rental rate, r^*, is somewhat higher than in the case of the Solow model due to the "reliability tax" on capital, whereas the net rental rate, $r_\dagger^*$, exactly equals the rental rate in the steady state of the Solow model.

Evaluating equations 17-13 and 17-14 using our earlier estimates of the quantities involved, yields:

$$r^* = \frac{31\%}{20\%} \times (6.3\% + 20\% \times 0.64\%) = 9.96\%$$

$$r_\dagger^* = \frac{31\%}{20\%} \times 6.3\% = 9.77\%$$

So, although, r^* is larger than $r_\dagger^*$, the practical difference between these two rental rates is relatively small.

17.5 THE CAPITAL INTENSITY

Using our equation for the rental rate, the previous result for r^* can be rewritten as:

$$(1 - \theta)v^* f_k(k^*) = \frac{\beta}{s}(\delta + g + s\tau). \qquad \text{17-15}$$

From what we know about the value of v^*, it follows that:

$$f_k(k^*) = \frac{\beta(\delta + g + s\tau)(\delta + g + \phi + \varepsilon)}{s(1 - \theta)(\delta + g + \phi)} \qquad \text{17-16}$$

Furthermore, since $f_k(\cdot)$ has an inverse $f_k^{-1}(\cdot)$, it follows that the economy's capital intensity, in equilibrium, becomes:

$$k^* = \left[\frac{sA(1-\theta)(\delta+g+\phi)}{(\delta+g+s\tau)(\delta+g+\phi+\varepsilon)}\right]^{1/\alpha} \quad \text{17-17}$$

Note that we now have an expression for k^* that is solely in terms of the basic parameters of the model. Since $1 > \alpha > 0$, there is a positive relationship between k^* and the term in square brackets. Consequently, any change to one of the parameters that increases (or decreases) the term in square brackets also increases (or decreases) k^*. For example, an increase in ε will necessarily decrease k^*, as would an increase in θ.

Writing k^* in terms of the basic parameters helps us conduct comparative statics but the complexity obscures much of the underlying economics. So, we conclude this section by recasting the expression in terms of more conventional quantities. Using what we know about v^*, our somewhat complex expression for k^* can be written rather more compactly as follows:

$$k^* = \left[\frac{sA(1-\theta)v^*}{\delta+g+s\tau}\right]^{1/\alpha}$$

In terms of the effective TFP and the rental rate, this then becomes:

$$k^* = \left(\frac{\beta B^*}{r^*}\right)^{1/\alpha},$$

where $B^* = A(1-\theta)v^*$ and $r^* = (\beta/s)(\delta+g+s\tau)$. This shows that the equilibrium capital intensity ultimately depends on the equilibrium values of effective TFP and the rental rate. Whilst r^* is fundamentally fixed by the most basic macroeconomic parameters, B^* depends on the regret rate, θ, and the reliability of the production process, v^*, and both these quantities are somewhat responsive to the performance of the national quality infrastructure.

17.6 THE CIRCULAR FLOW OF MONEY

Since $r^* = (1-\theta)v^* f_k(k^*)$ and $f_k(k^*) = \beta f(k^*)/k^*$, it follows that: $r^* = \beta(1-\theta)v^* f(k^*)/k^*$. Hence, a citizen's the rental income, in equilibrium, is given by:

$$r^*k^* = \beta(1-\theta)v^* f(k^*).$$

But, since $y^* = (1-\theta)v^* f(k^*)$, this expression for the per capita rental income can be written more compactly as follows:

$$r^*k^* = \beta y^*. \quad \text{17-18}$$

Using the equilibrium condition for $\dot{k}(t) = 0$ to substitute for r^*, yields the following expression: $(\beta/s)(\delta+g+s\tau)k^* = \beta y^*$. Cancelling β from both sides along with some rearrangement then gives an expression for a citizen's equilibrium savings:

$$sy^* = (\delta+g+s\tau)k^*. \quad \text{17-19}$$

Notice that this is a version of the circular flow equation, which says that gross savings equal gross investment. A little further rearrangement, yields:

$$s(y^* - \tau k^*) = (\delta+g)k^*. \quad \text{17-20}$$

Finally, using $y_\dagger^* = y^* - \tau k^*$, we arrive at a more compact version of the classic "circular flow" equation:

$$sy_\dagger^* = (\delta+g)k^*, \quad \text{17-21}$$

where the LHS of this expression is per capita savings, and the RHS is gross investment per capita, in the system's steady state. This equation must be satisfied for the circular flow of money to be in equilibrium: for savings to equal investment.

This additional interpretation of the equilibrium condition for $\dot{k}(t) = 0$ motivates the following diagram for the steady state of the capital stock.

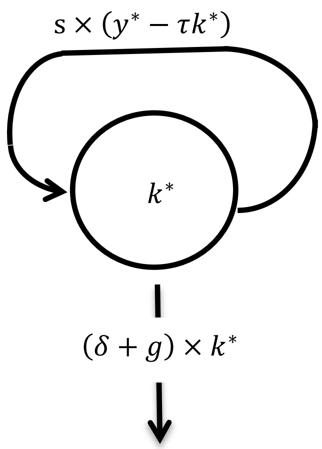

Figure 4: Capital intensity in the steady state

Note that because $y^* = (1 - \theta)v^* f(k^*)$, we can interpret this diagram as representing the steady state of a kind of Markov process. (Recall that v^* is a function of the model's basic parameters and does not depend on k^*.) Moreover, the arrows in the diagram relate to several economically meaningful things:

- The top arrow (that lopes round on itself) represents the savings taken from the net revenue. The total revenue from production is y^*, but we need to subtract τk^* from y^* because the cost of conformance testing eats into the income that is available for investment.
- Because of growth in the workforce (g) and depreciation (δ), the yearly decline in the amount of (existing) capital per worker must equal $(\delta + g)k(t)$, where the bracketed expression, $\delta + g$, denotes the gross replacement rate.[37] Hence, the bottom arrow represents two factors that tend to decrease capital intensity, and so this needs to be offset by an inflow of new capital from investment.

To summarise, there are economic factors that cause the capital per worker to decrease and others that cause it to increase. The bottom arrow represents an outflow due to depreciation and growth in the workforce, whereas the circular arrow, at the top of the diagram, represents the inflow of investment due to savings. The system is in a steady state when the stock of capital has reached a level where the outflow is balanced by the inflow, and this condition determines the equilibrium.

17.7 THE CAPITAL RATIO IN THE STEADY STATE

As already discussed, the capital ratio is the value of the capital stock normalised by the economy's output: k^*/y^*. The aim of this subsection is to find an expression for the capital ratio in terms of the basic parameters of the model.

[37] The workforce grows as the population itself grows, and so ever more capital is needed to maintain the same amount of capital per worker. This is why g features in the formula for the gross investment rate.

As already discussed, the "circular flow" equation is $sy^*_\dagger = (\delta + g)k^*$, where the net revenue is $y^*_\dagger = y^* - \tau k^*$. Thus, by eliminating $y^*_\dagger$ from this pair of equations, the economy's capital ratio can be written as follows:

$$\frac{k^*}{y^*} = \frac{s}{\delta + g + s\tau}. \qquad 17\text{-}22$$

Notice that the LHS is the economy's capital ratio, in equilibrium, and the RHS is solely in terms of the basic parameters of the model. Recall that $\delta + g$ is the gross investment rate, s is the savings rate, and τ is the cost of supervising one million pounds of capital. Hence, the formula can be written in words as follows:

$$\frac{\text{Capital}}{\text{GVA}} = \frac{(\text{savings rate})}{(\text{gross investment rate}) + (\text{savings rate}) \times (\text{supervision cost})}$$

From the "portion size" and wage rate, we found that for each one million pounds of capital there is an associated £6.4 thousand in supervision costs. Next, from the depreciation rate, δ, and growth rate, g, we found that the gross investment rate is 6.3%. Finally, it's already been established that the savings rate is 20%. Substituting these values into our formula gives:

$$\frac{\text{Capital}}{\text{GVA}} = \frac{20\%}{6.3\% + 20\% \times 0.64\%} = 311\%$$

Thus, evaluating this expression using the parameter values implies that the wealth contained in the capital stock is worth a bit over three times the yearly output of the economy.

17.8 SPENDING ON CONFORMANCE TESTS IN EQUILIBRIUM

Let m^* denote the cost of conformance testing as a proportion of the revenue from production. It follows that $m^* = \tau k^*/y^*$, where τk^* is the cost of conformance testing (payments to engineers) and y^* is the revenue from production (GVA).

Since τ is a fixed parameter, m^* is proportional to the economy's capital ratio:

$$m^* = \tau \times (k^*/y^*). \qquad 17\text{-}23$$

Using the values for τ and k^*/y^* from the previous subsection, this then yields:

$$\frac{\text{Cost}}{\text{GVA}} = 0.64\% \times 311\% = 2.0\%.$$

So, evaluating this expression using the parameter values implies that around 2% of output is spent on conformance testing. Furthermore, it's already been established that, in the steady state, the capital ratio becomes: $k^*/y^* = s/(\delta + g + s\tau)$. Which then implies that m^* can be written as follows:

$$m^* = \frac{s\tau}{\delta + g + s\tau}. \qquad 17\text{-}24$$

This gives us an expression for the proportion of revenue spent on conformance testing in terms of the basic parameters of the model.

17.9 THE PROSPERITY OF CITIZENS IN EQUILIBRIUM

Let c^* denote the per capita consumption in the steady state. Meaning that the consumption of the representative citizen in economy's equilibrium is given by:

$$c^* = (1 - s)y^*_\dagger, \qquad 17\text{-}25$$

where s is the savings rate. As already discussed, the "circular flow" equation (saying that savings equal investment) is $sy_{\dagger}^{*} = (\delta + g)k^{*}$. Thus, by eliminating $y_{\dagger}^{*}$ from this pair of equations, the equilibrium level of consumption can be written as follows:

$$c^{*} = \frac{1}{s}(1 - s)(\delta + g)k^{*}. \qquad \text{17-26}$$

Notice that the per capita consumption (prosperity) is an increasing function of k^{*}, and the factor multiplying k^{*} is composed solely of fundamental constants. Hence, anything that increases the capital intensity (such as, engineers getting better at finding, and fixing, malfunctioning machines) also increases peoples' living standards.

Finally, this previous equation could also be written as:

$$\frac{sc^{*}}{(1 - s)} = (\delta + g)k^{*}.$$

Notice that the RHS is the gross investment. The intuition behind this formula is that consumption divided by $1 - s$ gives the net-revenue. And, multiplying the net-revenue by the savings rate, s, gives the gross investment.

This concludes Part 2 of this report. The next step is to analyse the influence of the model's parameters on the system's equilibrium.

18 THE CURVATURE AND ELASTICITY OF A FUNCTION

This section begins Part 3 of this report, which uses techniques from the field of comparative statics to explore the effect of changes in the frequency of inspections on the equilibrium capital intensity. Specifically, Part 3 is focussed on deriving a first-order condition for the optimal inspection frequency, and thereby endogenises the amount spent by businesses on conformance testing. However, the first step is to establish some identities linking the curvature and elasticity of a function. Later, these identities will be used to prove that our expression for equilibrium capital intensity is quasi-concave with respect to the frequency of inspections.

This section introduces notation that helps to define the curvature and elasticity of a function. And, as discussed, it also establishes an identity for the elasticity of an elasticity that connects a function's curvature to its elasticity.

18.1 NOTATION FOR ELASTICITIES

This subsection defines $\mathcal{E}(\cdot)$ as a function that can be applied to a quantity that depends on a variable. Let this be referred to as the elasticity function.

Definition. *Suppose u is a quantity (e.g., utility) that depends on a variable x (e.g., consumption). Let $\mathcal{E}(u, x)$ denote the elasticity of u with respect to changes in x; where this elasticity is defined as follows:*

$$\mathcal{E}(u, x) := \frac{x}{u}\frac{\partial u}{\partial x} \qquad \text{18-1}$$

Let $\Delta u/u$ and $\Delta x/x$ denote proportional changes in u and x, respectively. In terms of these proportional changes, the elasticity can be approximated as follows:

$$\mathcal{E}(u, x) \approx \frac{\Delta u/u}{\Delta x/x} \qquad \text{18-2}$$

Thus, from equation 18-2, the elasticity is the proportional change in u divided by the proportional change in x. That is, $\mathcal{E}(u, x)$ is the proportional change in u that would occur if x were to double.

Note that the elasticity function can be applied in any situation where a quantity depends on a variable. The concept is not restricted to analysing the change in utility from variations in consumption.

18.2 A METRIC FOR THE CURVATURE OF A FUNCTION

This subsection defines $\mathcal{R}(\cdot)$ as a function that can be applied to a quantity u (e.g., utility) that depends on a variable x (e.g. consumption).

Definition. *Let $\mathcal{R}(\cdot)$ be referred to as the curvature function; where, for a quantity u and variable x, $\mathcal{R}(u, x)$ is defined as follows:*

$$\mathcal{R}(u, x) := -x\left(\frac{\partial^2 u/\partial x^2}{\partial u/\partial x}\right) \qquad 18\text{-}3$$

Traditionally, equation 18-3 is applied to utility functions and is interpreted as the coefficient of relative risk aversion (RRA). However, in the context of this study, $\mathcal{R}(\cdot)$ should be understood as a function that can be applied in any situation where there's a quantity that depends on variable. That is, it can be used as a general metric for a function's curvature in relation to changes in a particular variable.

18.3 THE ELASTICITY OF AN ELASTICITY

This subsection considers the partial derivative of $\mathcal{E}(u, x)$ with respect to x, from which one can find an expression for the "elasticity of an elasticity" based on the elasticity and curvature functions.

Lemma 18-1. *The elasticity function $\boldsymbol{\mathcal{E}}(\cdot)$ was defined by 18-1; and the curvature function $\boldsymbol{\mathcal{R}}(\cdot)$ was defined by 18-3. For a quantity $\boldsymbol{u}$ that depends on a variable $\boldsymbol{x}$, it can be shown that $\boldsymbol{\mathcal{E}}(\cdot)$ and $\boldsymbol{\mathcal{R}}(\cdot)$ are always related through the following identity:*

$$\boldsymbol{\mathcal{R}(u, x) + \mathcal{E}(u, x) + \mathcal{E}(\mathcal{E}(u, x), x) = 1}, \qquad 18\text{-}4$$

where

$$\mathcal{E}(\mathcal{E}(\cdot), x) := \frac{x}{\mathcal{E}(\cdot)}\frac{\partial \mathcal{E}(\cdot)}{\partial x}.$$

That is, $\mathcal{E}(\mathcal{E}(u, x), x)$ represents the "elasticity of an elasticity".

The proof of this lemma runs as follows:

Proof. Combining the basic definition of $\mathcal{E}(u, x)$ (from equation 18-1) with the chain-rule of differentiation, it follows that:

$$\frac{\partial \mathcal{E}(u, x)}{\partial x} = \frac{\partial (x/u)}{\partial x}\frac{\partial u}{\partial x} + (x/u)\frac{\partial^2 u}{\partial x^2}$$

Applying the quotient-rule to the derivative of x/u, gives:

$$\frac{\partial \mathcal{E}(u, x)}{\partial x} = \left(u - x\frac{\partial u}{\partial x}\right)\frac{1}{u^2}\frac{\partial u}{\partial x} + \frac{x}{u}\frac{\partial^2 u}{\partial x^2}$$

With a little rearrangement, this can be rewritten as follows:

$$\frac{\partial \mathcal{E}(u, x)}{\partial x} = \left(1 - \frac{x}{u}\frac{\partial u}{\partial x}\right)\frac{1}{u}\frac{\partial u}{\partial x} + \frac{x}{u}\frac{\partial^2 u}{\partial x^2}$$

Multiply both sides of this expression by x to get:

$$x\frac{\partial \mathcal{E}(u, x)}{\partial x} = \left(1 - \frac{x}{u}\frac{\partial u}{\partial x}\right)\frac{x}{u}\frac{\partial u}{\partial x} + \frac{x^2}{u}\frac{\partial^2 u}{\partial x^2}$$

Using this basic definition of $\mathcal{E}(u, x)$ this can be written as follows:

$$x\frac{\partial \mathcal{E}(u,x)}{\partial x} = [1-\mathcal{E}(u,x)]\mathcal{E}(u,x) + \frac{x^2}{u}\frac{\partial^2 u}{\partial x^2}$$

Dividing both sides of this expression through by $\mathcal{E}(u,x)$ gives the main result. With a little further rearrangement this concludes the proof.■

18.4 COMPARATIVE STATICS

In this study we take the most fundamental parameters of the economy as given. That is, δ, g and s are regarded a rigidly fixed. In addition, the rate at which malfunctions occur, ε, is also be regarded as being fixed by the existing technology. However, we will suppose that society has some control over the parameters that govern the conformance testing regime. That is, over the long term, society can make decisions about the value of these parameters.

The system's key variables are:

- The proportion of output spent on conformance testing, m.
- The reliability of production, v.
- The economy's capital intensity, k.

Subsequent sections explore the relationship between the equilibrium values of these variables and the parameters governing the CT regime. That is, it looks at how changes in these parameters would feed through to changes in the equilibrium values of the system's variables: m^*, v^*, and k^*. This kind of analysis will be referred to as "comparative statics".

These sections have two specific areas of interest: Firstly, it conducts an analysis for the three parameters that depend on the frequency of inspections: ϕ, θ, and τ. In the next section, these results will be used to characterise an optimal frequency. Secondly, it looks at the effect of changes in the frequency of inspections, n, on capital intensity, k^*. Later on, these results are used to explore the macroeconomic impact of businesses losing access to high-quality calibration services.

19 COMPARATIVE STATICS FOR THE FREQUENCY OF INSPECTIONS

This subsection gives some intermediate results that are used throughout the rest of the section. Recall that θ, ϕ, and τ are defined as follows:

$$\theta = 1-\exp(-p_{1|0}n)\,;\ \phi = 1-\exp(-p_{1|1}n)\,; \tau = (\omega/a)n$$

Notice that these are all functions of the frequency of inspections, n. The corresponding partial derivatives are:

$$\frac{\partial\theta}{\partial n} = p_{1|0}(1-\theta);\ \frac{\partial\phi}{\partial n} = p_{1|1}(1-\phi);\ \frac{\partial\tau}{\partial n} = \frac{\tau}{n}.$$

In terms of the corresponding elasticities, we have:

$$\mathcal{E}(\theta,n) = n p_{1|0}\,(1-\theta)/\theta\,; \quad 19\text{-}1$$

$$\mathcal{E}(\phi,n) = n p_{1|1}\,(1-\phi)/\phi\,; \quad 19\text{-}2$$

$$\mathcal{E}(\tau,n) = 1\ . \quad 19\text{-}3$$

These results will be used repeatedly in the following subsections.

Next, we use these results to find an expression for the elasticity of m^* with respect to variations in n. That is, it shows how variations in the frequency of inspections relate to changes in the proportion of the economy's output spent on conformance testing.

Lemma 19-1. $\boldsymbol{m}^*$ *denotes the proportion of revenue spent on conformance testing in the system's steady state. The elasticity of* $\mathbf{m}^*$ *with respect to the frequency of inspections* $\boldsymbol{n}$ *just becomes:*

$$\mathcal{E}(m^*, n) = 1 - m^* \qquad 19\text{-}4$$

The proof runs as follows:

Proof. Taking logs of the expression for m^* and differentiating with respect to n gives:

$$\frac{1}{m^*}\frac{\partial m^*}{\partial n} = \frac{1}{\tau}\frac{\partial \tau}{\partial n} - \frac{s}{(\delta + g + s\tau)}\frac{\partial \tau}{\partial n}$$

Multiplying both sides through by τ yields:

$$\frac{\tau}{m^*}\frac{\partial m^*}{\partial n} = \frac{\partial \tau}{\partial n} - \frac{s\tau}{(\delta + g + s\tau)}\frac{\partial \tau}{\partial n}$$

From the equation for m^*, this then becomes:

$$\frac{\tau}{m^*}\frac{\partial m^*}{\partial n} = (1 - m^*)\frac{\partial \tau}{\partial n}$$

Multiply both sides through by n/τ to get:

$$\frac{n}{m^*}\frac{\partial m^*}{\partial n} = (1 - m^*)\frac{n}{\tau}\frac{\partial \tau}{\partial n}$$

In terms of our notation for elasticities this can be rewritten as:

$$\mathcal{E}(m^*, n) = (1 - m^*)\mathcal{E}(\tau, n).$$

Lastly, τ is proportional to n, implying that $\mathcal{E}(\tau, n) = 1$; which concludes the proof. ■

20 COMPARATIVE STATICS FOR THE RELIABILITY OF PRODUCTION

This subsection finds an expression for how varying the frequency of inspections effects the reliability of the economy's production process.

The detection rate, ϕ, depends directly on n, and so v^* inherits a dependence on n through its connection to ϕ. Therefore, if n varies whilst the other determinants of v^* remain fixed, then v^* becomes a function of n. If v^* was plotted against n, then this relationship could be represented as a smooth curve with v^* on the vertical axis and n on the horizontal axis. Such a curve will have both an elasticity $\mathcal{E}(v^*, n)$ and a curvature that varies as n varies.

Figure 3 can help us to work through what happens when n increases: If the frequency of inspections where to increase, then there's an increase in the detection rate. That is, if $n \rightarrow n + \Delta n$, then $\phi \rightarrow \phi + \Delta\phi$. In terms of Figure 3, this increases the flow of probability out of the "bad" bucket and into the "good" bucket. (That is, increasing the frequency of inspections stregthens the horizontal pointing from right to left.) Consequently, the equilibrium will adjust such that $v^* \rightarrow v^* + \Delta v^*$, and thus the reliability of production will increase.

These considerations lead to a lemma connecting the ealsticity of v^* to the elasticity of ϕ.

Lemma 20-1. $\boldsymbol{v}^*$*denotes the reliability of the economy's production processes in equilibrium; and* $\boldsymbol{n}$ *is the frequency of inspections under a conformance testing regime.* $\boldsymbol{\mathcal{E}}(\boldsymbol{v}^*, \boldsymbol{n})$ *and* $\boldsymbol{\mathcal{E}}(\boldsymbol{\phi}, \boldsymbol{n})$ *denote the elasticity of* $\boldsymbol{v}^*$ *and* $\boldsymbol{\phi}$ *with respect to* $\boldsymbol{n}$*. The ratio of these two elasticities is given by the following formula:*

$$\frac{\mathcal{E}(v^*, n)}{\mathcal{E}(\phi, n)} = \frac{\phi(1 - v^*)}{\delta + g + \phi} \qquad 20\text{-}1$$

Since $0 < v^* < 1$ and $\mathcal{E}(\phi, n)$ is positive, this implies $\mathcal{E}(v^*, n) > 0$; and so, the reliability of production rises, v^*, as the frequency of inspections increases, n.

Consider the RHS of equation 20-1. The denominator is the proportion of malfunctioning machines that are either fixed or that exit the stock naturally due to depreciation. That is, the denominator represents the proportional outflow of probability from $1 - v^*$ when the system is in its steady state.

Next, recall that $1 - v^*$ is the proportion of machines that will continue to malfunction given a certain frequency of inspections; and ϕ is the likelihood of catching such a malfunction with this number of inspections. Hence, the numerator of equation 20-1 represents the increase in the detection of malfunctioning machines that would occur if the frequency of inspections were to double: $n \to 2n$. That is, the numerator is the change in the flow of probability into v^* that occurs if the inspection rate were to double.[38]

The proof of equation 20-1 runs as follows:

Proof. The chain-rule of differentiation gives:

$$\frac{\partial v^*}{\partial n} = \frac{\partial v^*}{\partial \phi}\frac{\partial \phi}{\partial n}$$

Now, multiply through by n/v^* to get an expression for the elasticity:

$$\mathcal{E}(v^*, n) = \frac{n}{v^*}\frac{\partial v^*}{\partial \phi}\frac{\partial \phi}{\partial n}$$

The RHS of this expression can then be rewritten as:

$$\mathcal{E}(v^*, n) = \mathcal{E}(v^*, \phi)\mathcal{E}(\phi, n)$$

An expression for $\mathcal{E}(v^*, \phi)$ can be found as follows: Taking logs of v^* and differentiating with respect to ϕ gives:

$$\frac{1}{v^*}\frac{\partial v^*}{\partial \phi} = \varepsilon(\delta + g + \phi + \varepsilon)^{-1}(\delta + g + \phi)^{-1}$$

Multiplying through by ϕ leads to the following result:

$$\mathcal{E}(v^*, \phi) = \varepsilon\phi(\delta + g + \phi + \varepsilon)^{-1}(\delta + g + \phi)^{-1}$$

Substituting this result into the earlier expression for $\mathcal{E}(v^*, n)$ then yields:

$$\mathcal{E}(v^*, n) = \varepsilon\phi(\delta + g + \phi + \varepsilon)^{-1}(\delta + g + \phi)^{-1}\mathcal{E}(\phi, n)$$

Finally, recalling that $1 - v^* = \varepsilon(\delta + g + \phi + \varepsilon)^{-1}$, gives the main result.■

Lemma 20-2. *$\boldsymbol{v}^*$ denotes the reliability of production processes in equilibrium, and $\boldsymbol{n}$ is the frequency of inspections under a conformance testing regime. Moreover, $\boldsymbol{v}^*$ inherits a dependence on $\boldsymbol{n}$ through its connection to the detection rate, $\boldsymbol{\phi}$. The curvature metric $\boldsymbol{\mathcal{R}}(\cdot)$ is defined by 18-3, and $\boldsymbol{\mathcal{R}}(\boldsymbol{v}^*, \boldsymbol{n})$ represents the curvature of the function that's generated by plotting $\boldsymbol{v}^*$ against $\boldsymbol{n}$.*

(a) $\mathcal{R}(v^*, n)$ is given by the following formula:

[38] The rebate rate is $(1 - \phi)(1 - v^*)$ and the frequency of inspections is n. Now, consider what happens if the frequence of inspections doubles: Since the power of a test is $p_{1|1}$, the increase in the detection rate from another round of inspections is $p_{1|1}n \times (1 - \phi)(1 - v^*)$.

$$\boldsymbol{\mathcal{R}(v^*, n)} = \frac{\boldsymbol{\phi}}{\boldsymbol{1-\phi}}\boldsymbol{\mathcal{E}(\phi, n)} + \frac{\boldsymbol{2v^*}}{\boldsymbol{1-v^*}}\boldsymbol{\mathcal{E}(v^*, n)}, \qquad \text{20-2}$$

where $\boldsymbol{\mathcal{E}(\phi, n)}$ is the elasticity of $\boldsymbol{\phi}$ with respect to $\boldsymbol{n}$; and $\boldsymbol{\mathcal{E}(v^*, n)}$ is the elasticity of $\boldsymbol{v^*}$ with respect to $\boldsymbol{n}$.

(b) Both the elasticities in 20-2 are positive and $\boldsymbol{v^*, \phi}$ belong to the unit interval, implying that: $\boldsymbol{\mathcal{R}(v^*, n) > 0}$.

(c) v^* is an increasing, concave function of n, meaning: $\partial v^*/\partial n > 0$; and $\partial^2 v^*/\partial n^2 < 0$.

The proof of this lemma runs as follows:

Proof. From 20-1, we have:

$$\mathcal{E}(v^*, n) = \frac{\phi\mathcal{E}(\phi, n)(1 - v^*)}{\delta + g + \phi}\ \text{, where } \mathcal{E}(v^*, n) := \frac{n}{v^*}\frac{\partial v^*}{\partial n}.$$

A little rearrangement then gives:

$$\frac{\partial v^*}{\partial n} = \frac{\phi\mathcal{E}(\phi, n)(1 - v^*)v^*}{n(\delta + g + \phi)}$$

By using 19-2 to substitute for $\mathcal{E}(\phi, n)$, this can be rewritten as:

$$\frac{\partial v^*}{\partial n} = \frac{p_{1|1}(1 - \phi)(1 - v^*)v^*}{\delta + g + \phi}$$

Taking logs of our previous expression for $\partial v^*/\partial n$ gives

$$\ln(\partial v^*/\partial n) = \ln(p_{1|1}) + \ln(1 - \phi) + \ln(1 - v^*) + \ln(v^*) - \ln(\delta + g + \phi)$$

Then, differentiating this expression with respect to n yields:

$$\frac{\partial^2 v^*/\partial n^2}{\partial v^*/\partial n} = -\frac{\partial\phi/\partial n}{1 - \phi} - \frac{\partial v^*/\partial n}{1 - v^*} + \frac{\partial v^*/\partial n}{v^*} - \frac{\partial\phi/\partial n}{\delta + g + \phi}$$

Multiply through by n to get:

$$n\left(\frac{\partial^2 v^*/\partial n^2}{\partial v^*/\partial n}\right) = -\frac{\phi}{1 - \phi}\left(\frac{n}{\phi}\frac{\partial\phi}{\partial n}\right) - \frac{v^*}{1 - v^*}\left(\frac{n}{v^*}\frac{\partial v^*}{\partial n}\right) + \left(\frac{n}{v^*}\frac{\partial v^*}{\partial n}\right) - \frac{\phi}{\delta + g + \phi}\left(\frac{n}{\phi}\frac{\partial\phi}{\partial n}\right)$$

Use 18-1 and 18-3 to rewrite the expression as follows:

$$-\mathcal{R}(v^*, n) = -\frac{\phi}{1 - \phi}\mathcal{E}(\phi, n) - \frac{v^*}{1 - v^*}\mathcal{E}(v^*, n) + \mathcal{E}(v^*, n) - \frac{\phi}{\delta + g + \phi}\mathcal{E}(\phi, n)$$

Use 20-1 to substitute for the last term in the previous expression:

$$-\mathcal{R}(v^*, n) = -\frac{\phi}{1 - \phi}\mathcal{E}(\phi, n) - \frac{v^*}{1 - v^*}\mathcal{E}(v^*, n) + \mathcal{E}(v^*, n) - \frac{1}{1 - v^*}\mathcal{E}(v^*, n)$$

Next, gather all the terms containing $\mathcal{E}(v^*, n)$ to get:

$$-\mathcal{R}(v^*, n) = -\frac{\phi}{1 - \phi}\mathcal{E}(\phi, n) - \frac{2v^*}{1 - v^*}\mathcal{E}(v^*, n)$$

Multiply through by -1 to get the main result. Furthermore, since ϕ, v^* belong to the unit interval, and the elasticities are positive, it must be the case that $\mathcal{R}(v^*, n) > 0$. Lastly, since v^* is an increasing function of n, it follows that $\partial^2 v^*/\partial n^2 < 0$.■

21 COMPARATIVE STATICS FOR CAPITAL INTENSITY

$v^* \in (0,1)$ is the reliability of production processes in equilibrium. That is, v^* is the fraction of total output that is "good", and thus usable. With perfect production processes (i.e. $v^* = 1$) there would be no such thing as "bad" outputs; meaning that there would be no need for conformance testing, and so no losses from type-1 errors. In this ideal world, the usable output from each worker would be $f(k^*)$, where $f(\cdot)$ is the intensive form of the production function and k^* is the economy's capital intensity in equilibrium. In such a situation, the marginal product of capital (MPK) would be $f_k(k^*)$, where $f_k(\cdot)$ is the first derivative of $f(\cdot)$ evaluated at k^*.

However, in the real world, the MPK is inevitably reduced by using less than perfect production processes, along with type-1 errors in conformance testing. Consequently, the MPK becomes $(1-\theta)v^* f_k(k^*)$, where $\theta \in (0,1)$ is the false positive rate, and $1-\theta$ is the loss from occasionally rejecting "good" output.

r^* denotes the rental rate in the system's equilibrium. Competitive factor markets, and constant returns-to-scale, ensure that the rental rate, r^*, equals the MPK, and this implies that $r^* = (1-\theta)v^* f_k(k^*)$. Lastly, it's already been shown that a version of Piketty's formula implies that $r^* = (\beta/s)(\delta + g + s\tau)$, where the RHS features the model's basic parameters. Bringing these two expressions for r^* together leads to the following proposition:

Proposition 21-1*.* $\boldsymbol{v}^*$ *and* $\boldsymbol{k}^*$ *are, respectively, the reliability of production processes and capital intensity in the system's equilibrium. Equating two distinct formulae - one for the marginal product of capital (MPK) and another for the rental rate - yields the following equilibrium relationship:*

$$(1-\theta)v^* f_k(k^*) = (\beta/s)(\delta + g + s\tau). \qquad \text{21-1}$$

The lefthand and righthand sides of 21-1 should be interpreted as follows:

(a) *The LHS of 21-1 is the MPK, and is composed of the following elements:* v^**is the faction of total output that is "good" in the sense of being usable;* θ *is the proportion of "good" outputs mistakenly rejected during conformance testing; and* $f_k(k^*)$ *represents the MPK in a world with perfect production processes.*

(b) *The RHS of 21-1 is the rental rate, and features the following elements:* β *is capital's share of output;* s *is the savings rate;* δ *is the depreciation rate;* g *is a growth rate of workforce; and* τ *is the rate at which capital is "taxed" to pay for conformance testing.*

Lemma 21-1*.* $\boldsymbol{n}$ *is the frequency of inspections under a conformance testing regime, and* $\boldsymbol{p_{1|0}}$ *is the likelihood that output from a production plant falls prey to type-1 errors.* $\boldsymbol{v}^*$ *and* $\boldsymbol{k}^*$ *are, respectively, the reliability of production processes and capital intensity in the system's equilibrium.* $\boldsymbol{v}^*, \boldsymbol{k}^*$ *depend on the frequency of testing,* $\boldsymbol{n}$*, where* $\boldsymbol{\mathcal{E}(v^*, n)}$ *and* $\boldsymbol{\mathcal{E}(k^*, n)}$ *are the elasticities of* $\boldsymbol{v}^*$ *and* $\boldsymbol{k}^*$*, respectively. It can be shown that the elasticity of* $\boldsymbol{k}^*$ *with respect to* $\boldsymbol{n}$ *is given by the following formula:*

$$\mathcal{E}(k^*, n) = \frac{1}{\alpha}\left[\mathcal{E}(v^*, n) - p_{1|0}n - m^*\right], \qquad \text{21-2}$$

where m^* *is the proportion of output spent on employing CT engineers; and* α *is the proportion of output going to labour in the form of wages.*

What is the intuition behind our equation for $\mathcal{E}(k^*, n)$? This elasticity represents the percentage change in the economy's equilibrium capital intensity, k^*, that would result from doubling the frequency of inspections, n. In other words, $\mathcal{E}(k^*, n)$ is the proportional change in k^* that occurs over a period of years when the system transitions from an initial steady state involving n inspections per year to a new steady state with $2n$ inspections per year. The RHS of the equation is composed of three terms:

- $\mathcal{E}(v^*, n)$ represents the percentage increase in the reliability, v^*, of production that would come from doubling the frequency of inspections, n.
- $p_{1|0}n$ is the frequency with which type-1 errors cause viable output to be mistakenly scrapped. Some of the lost output would have gone into investment, and so the increase in such mistakes tends to slightly reduce the economy's capital intensity.
- m^* is the cost of conformance testing in the steady state of the system. The payments to CT engineers consume a little of the output that would otherwise be used for investment in new capital equipment.

The proof of Lemma 21-1 runs as follows:

Proof. According to Proposition 21-1, v^* and k^* are connected through equation 21-1. Taking logs of 21-1 gives the following expression:

$$\ln(1-\theta) + \ln(v^*) + \ln[f_k(k^*)] = \ln(\beta/s) + \ln(\delta + g + s\tau).$$

Next, recall that the parameters ϕ, θ, and τ all depend on the inspection frequency, n. The relationship between k^* and n can be found by differentiating the previous expression:

$$\frac{f_{kk}(k^*)}{f_k(k^*)}\frac{\partial k^*}{\partial n} + \frac{1}{v^*}\frac{\partial v^*}{\partial n} - \frac{1}{(1-\theta)}\frac{\partial \theta}{\partial n} = s(\delta + g + s\tau)^{-1}\frac{\partial \tau}{\partial n}$$

Since $f_{kk}(k^*)/f_k(k^*) = -\alpha/k^*$, this then becomes:

$$-\frac{\alpha}{k^*}\frac{\partial k^*}{\partial n} + \frac{1}{v^*}\frac{\partial v^*}{\partial n} - \frac{1}{(1-\theta)}\frac{\partial \theta}{\partial n} = s(\delta + g + s\tau)^{-1}\frac{\partial \tau}{\partial n}$$

Differentiating θ and τ with respect to n yields $\partial\theta/\partial n = p_{1|0}(1-\theta)$ and $\partial\tau/\partial n = \tau/n$. And, substituting these derivatives into the previous expression gives:

$$-\frac{\alpha}{k^*}\frac{\partial k^*}{\partial n} + \frac{1}{v^*}\frac{\partial v^*}{\partial n} - p_{1|0} = \frac{s\tau}{n}(\delta + g + s\tau)^{-1}$$

Multiplying both sides through by n then gives:

$$-\alpha\left(\frac{n}{k^*}\frac{\partial k^*}{\partial n}\right) + \left(\frac{n}{v^*}\frac{\partial v^*}{\partial n}\right) - p_{1|0}n = \frac{s\tau}{\delta + g + s\tau}$$

Furthermore, because $m^* = s\tau(\delta + g + s\tau)^{-1}$, this expression can be rewritten as follows:

$$-\alpha\left(\frac{n}{k^*}\frac{\partial k^*}{\partial n}\right) + \left(\frac{n}{v^*}\frac{\partial v^*}{\partial n}\right) - p_{1|0}n = m^*$$

A little further rearrangement gives us the main result. Lastly, recall that α is a fixed parameter and that $\mathcal{E}(v^*, n)$ is necessarily positive. Hence, the sign of $\mathcal{E}(k^*, n)$ is positive (negative) depending on whether $\mathcal{E}(v^*, n)$ is greater than (less than) the sum of $p_{1|0}n$ and m^*. ∎

Lemma 21-2: *$\boldsymbol{v}^*$ denotes the reliability of production processes in equilibrium; $\boldsymbol{k}^*$ denotes the capital intensity in equilibrium; and $\boldsymbol{n}$ is the frequency of inspections under a conformance testing regime. The elasticity function $\boldsymbol{\mathcal{E}}(\cdot)$ is defined by* **18-1**. *$\boldsymbol{\mathcal{E}}(\boldsymbol{v}^*, \boldsymbol{n})$ and $\boldsymbol{\mathcal{E}}(\boldsymbol{k}^*, \boldsymbol{n})$ are, respectively, the elasticities of $\boldsymbol{v}^*$ and $\boldsymbol{k}^*$ with respect to $\boldsymbol{n}$. The curvature metric $\boldsymbol{\mathcal{R}}(\cdot)$ is defined by 18-3, and $\boldsymbol{\mathcal{R}}(\boldsymbol{k}^*, \boldsymbol{n})$ represents the curvature of the function that's generated by plotting $\boldsymbol{k}^*$ against $\boldsymbol{n}$. It can be shown that $\boldsymbol{\mathcal{R}}(\boldsymbol{k}^*, \boldsymbol{n})$ is given by the following equation:*

$$\mathcal{R}(k^*,n) = \frac{1}{\alpha\mathcal{E}(k^*,n)}\left[n p_{1|1}\mathcal{E}(v^*,n) + \frac{1+v^*}{1-v^*}\{\mathcal{E}(v^*,n)\}^2 - \alpha\{\mathcal{E}(k^*,n)\}^2 - \{m^*\}^2\right] \quad 21\text{-}3$$

Lemma 21-2 can be proved as follows.

Proof. Differentiate 21-2 and then multiply through by n to get the following expression:

$$\alpha n \frac{\partial\mathcal{E}(k^*,n)}{\partial n} = n\frac{\partial\mathcal{E}(v^*,n)}{\partial n} - n p_{1|0} - n\frac{\partial m^*}{\partial n}$$

This can be rewritten as follows:

$$\alpha\mathcal{E}(k^*,n)\left[\frac{n}{\mathcal{E}(k^*,n)}\frac{\partial\mathcal{E}(k^*,n)}{\partial n}\right] = \mathcal{E}(v^*,n)\left[\frac{n}{\mathcal{E}(v^*,n)}\frac{\partial\mathcal{E}(v^*,n)}{\partial n}\right] - n p_{1|0} - m^*\left[\frac{n}{m^*}\frac{\partial m^*}{\partial n}\right]$$

Which, in neater notation, is equivalent to:

$$\alpha\mathcal{E}(k^*,n).\mathcal{E}(\mathcal{E}(k^*,n),n) = \mathcal{E}(v^*,n).\mathcal{E}(\mathcal{E}(v^*,n),n) - n p_{1|0} - m^*.\mathcal{E}(m^*,n)$$

Using 18-4 and 19-4, this then becomes:

$$\alpha\mathcal{E}(k^*,n)[1-\mathcal{E}(k^*,n)-\mathcal{R}(k^*,n)] = \mathcal{E}(v^*,n)[1-\mathcal{E}(v^*,n)-\mathcal{R}(v^*,n)] - n p_{1|0} - m^*(1-m^*)$$

Using 20-2 to substitute for $\mathcal{R}(v^*,n)$ gives:

$$\begin{aligned}\alpha\mathcal{E}(k^*,n)&[1-\mathcal{E}(k^*,n)-\mathcal{R}(k^*,n)]\\ &= \mathcal{E}(v^*,n)\left[1-\mathcal{E}(v^*,n)-\frac{\phi}{1-\phi}\mathcal{E}(\phi,n)-\frac{2v^*}{1-v^*}\mathcal{E}(v^*,n)\right] - n p_{1|0}\\ &- m^*(1-m^*)\end{aligned}$$

Simplifying the bracketed expression on the LHS gives:

$$\begin{aligned}\alpha\mathcal{E}(k^*,n)&[1-\mathcal{E}(k^*,n)-\mathcal{R}(k^*,n)]\\ &= \mathcal{E}(v^*,n)\left[1-\frac{\phi}{1-\phi}\mathcal{E}(\phi,n)-\frac{1+v^*}{1-v^*}\mathcal{E}(v^*,n)\right] - n p_{1|0} - m^*(1-m^*)\end{aligned}$$

Using 19-2 to substitute for $\mathcal{E}(\phi,n)$ gives:

$$\begin{aligned}\alpha\mathcal{E}(k^*,n)&[1-\mathcal{E}(k^*,n)-\mathcal{R}(k^*,n)]\\ &= \mathcal{E}(v^*,n)\left[1-n p_{1|1}-\frac{1+v^*}{1-v^*}\mathcal{E}(v^*,n)\right] - n p_{1|0} - m^*(1-m^*)\end{aligned}$$

Some rearrangement of the LHS gives:

$$\begin{aligned}\alpha\mathcal{E}(k^*,n)&[1-\mathcal{E}(k^*,n)-\mathcal{R}(k^*,n)]\\ &= \left[\mathcal{E}(v^*,n)-n p_{1|0}-m^*\right] + \{m^*\}^2 - n p_{1|1}\mathcal{E}(v^*,n) - \frac{1+v^*}{1-v^*}\{\mathcal{E}(v^*,n)\}^2\end{aligned}$$

Use 21-2 to substitute for the bracketed expression:

$$\alpha\mathcal{E}(k^*,n)[1-\mathcal{E}(k^*,n)-\mathcal{R}(k^*,n)] = \alpha\mathcal{E}(k^*,n) + \{m^*\}^2 - n p_{1|1}\mathcal{E}(v^*,n) - \frac{1+v^*}{1-v^*}\{\mathcal{E}(v^*,n)\}^2$$

A little simplification then yields:

$$-\alpha\mathcal{E}(k^*,n)[\mathcal{E}(k^*,n)+\mathcal{R}(k^*,n)] = \{m^*\}^2 - n p_{1|1}\mathcal{E}(v^*,n) - \frac{1+v^*}{1-v^*}\{\mathcal{E}(v^*,n)\}^2$$

Furthermore, this implies that:

$$-\alpha\mathcal{E}(k^*,n).\mathcal{R}(k^*,n) = \alpha\{\mathcal{E}(k^*,n)\}^2 + \{m^*\}^2 - n p_{1|1}\mathcal{E}(v^*,n) - \frac{1+v^*}{1-v^*}\{\mathcal{E}(v^*,n)\}^2$$

Multiplying through by -1 gives the main result, which concludes the proof. ■

22 THE OPTIMAL INSPECTION FREQUENCY

The nature of this equilibrium depends on the values of the basic parameters; and it has already been shown that there exists a positive relationship between the per capita consumption in an equilibrium and the capital intensity in equilibrium. In other words, the prosperity of citizens increases as the economy's capital intensity increases.

Until now, the frequency of inspections, n, has been regarded as another fixed parameter of the economy. However, in this section, we suppose that n can change in response to the optimising behaviour of millions of businesses. That is, until this point we have considered a relatively short span of time during which n is a fixed by existing routines, but in this section we consider a longer span of time, over which n can undergo change. Furthermore, what happens to n in the long run is determined by millions of businesses continually optimising their routines in an iterative manner.

We argue that buyers, sellers, and producers agree the frequency of inspections in the same way that they are able establish other norms and routines. Moreover, we presume that, through a combination of self-interested behaviour and market processes, a free society will find itself in an equilibrium that maximises the prosperity of all its citizens.

22.1 ENLIGHTENED SELF-INTEREST

Most of the model's parameters are fixed in ways that means they cannot be altered by peoples' behaviour or choices. For example, the depreciation rate is something that society must just take as given. However, society has control over the savings rate, s, and the frequency of inspections, n.

It's reasonable to suppose that a well-functioning society will select values for these parameters to maximise their long run prosperity of its citizens. Operationally, this means that society will select the values that maximises per capita consumption in the steady state: c^*. Moreover, it's already been shown that per capita consumption is proportional to the net revenue per worker: $y_\dagger^*$.

The value of the savings rate, s, is already assumed to be optimal in the sense that it maximises the populations' aggregate lifetime happiness by balancing their current and future consumption. More formally, the optimal savings rate corresponds to the solution of Ramsey's intertemporal optimisation problem, but this isn't the subject of our study and so it receives this rather cursory attention.

However, the frequency of inspections, n, is central to this study; and we argue that the economy will settle on a value of n that maximises the long run prosperity of all its citizens. Setting up a conceptual framework for characterising this optimum value of n is the focus of this section.

22.2 ITERATING TOWARDS THE "BEST" OF ALL POSSIBLE WORLDS

Let us assume that the economy settles into the steady-state equilibrium as detailed above. In this analysis, the frequency of inspections, n, was treated as one of the basic parameters of the economy. However, as explained above, n is better seen as a variable that can be chosen by society so that it picks out the best possible equilibrium.

The analysis in this section is premised on the assumption that, in the long run, a free and equal society generates the best possible outcome for its citizens. This is based on the theory that a blend market mechanisms and self-interested behaviour should lead to outcomes that serve the interests of the typical citizen. That is, the "invisible hand" should select the best possible outcome (i.e., efficient production and allocation) providing that people are prevented from accruing, or exerting, power in ways that interfere with market processes.

Suppose that economic and political processes are characterised by two things:

- Free entry and exit: There are no barriers to entry, and so incumbents are subject to active competition from rivals, or, at least, the threat of new entrants.
- Free choice: Citizens are left free to choose where and what they buy.

In this sort of environment, a kind of Darwinian selection ensures that what survives is what works best, where the detail of this argument runs as follows: Firstly, in a competitive market, the price for one unit of the good is bid down to its lowest possible marginal cost (achievable with the available technology). And, in this efficient situation, the lowest marginal cost will also equal the average unit cost of the good. Secondly, an inefficient firm is one that has a marginal cost above this minimum cost. Furthermore, because a firm can't influence the price of the good in the product market (or the factor prices for labour and capital), such a firm must make an economic loss. Therefore, an inefficient firm must either modify its routines so that it produces at the lowest possible cost, or it will go out of business. Either way, such inefficient firms cannot persist for long in a competitive market.

Following this line of argument, efficient production is underpinned by competitive markets, whilst allocations of the resulting outputs are aligned to peoples' preferences. So, put in its most naïve form, the implication is that we live in something that approximates to the "best" of all possible worlds. That is, the outcome is "optimal" within the constraints imposed by the economy's basic parameters.

More formally, this somewhat optimistic vision is grounded in one of the fundamental theories of classical economics: The First Welfare Theorem says that a perfectly competitive free-market economy will produce efficient outcomes (i.e., efficient production and allocation). Consequently, the economy ought to settle on a value of n that maximises the long run prosperity of its citizens. In other words, through a process of trial-and-improvement, the economy will select the value of n that maximises per capita consumption in the steady state; and let $\hat{n}$ denote this optimal value. Furthermore, since per capita consumption is proportional to k^*, it then follows that $\hat{n}$ corresponds to the value of n that maximises k^*.

Lastly, the abstract economic paradigm, described above, seems far removed from the less than perfect world we experience. So, let us acknowledge that whilst the world we inhabit doesn't live up to this ideal, our description of the paradigm brings out assumptions at the heart of classical economics, whose status is somewhat akin to those found Carnot Cycle of Thermodynamics. That is, although, not exactly true, the paradigm is true enough for model building and analysis.

22.3 QUASI-CONCAVITY AND THE EXISTENCE OF AN OPTIMAL FREQUENCY

The first step towards characterising the optimum inspection frequency is to find an expression for the effect of small changes in n on the economy's long-run capital intensity. In other words, we need an expression for the elasticity of k^* with respect to n. The next step is to use this expression to demonstrate that k^* is a quasi-concave function of n. The final step is to use the expression to find a first-order condition for the optimum value.

Lemma 22-1. *If the frequency of inspections, $\boldsymbol{n} \in \mathbb{R}_{++}$, is allowed to vary whilst all the other parameters remain constant, then the equilibrium capital intensity, $\boldsymbol{k}^*$, becomes a function of the inspection frequency: $\boldsymbol{k}^*(\boldsymbol{n})$. This function has the following properties:*

(a) *If $\boldsymbol{\partial k^*/\partial n = 0}$, then $\boldsymbol{\partial^2 k^*/\partial n^2 < 0}$; which implies that $\boldsymbol{k}^*(\boldsymbol{n})$ is concave in the vicinity of a stationary point.*

(b) *Any stationary point of $\boldsymbol{k}^*(\boldsymbol{n})$ has to be a local maximum, implying that there cannot exist a local minimum.*

(c) $\boldsymbol{k^*(n)}$ *is quasi-concave with a unique stationary point,* $\boldsymbol{\hat{n}}$*, such that* $\boldsymbol{\mathcal{E}[k^*(\hat{n}), \hat{n}] = 0}$*. Moreover, this stationary point must be the global maximum of* $\boldsymbol{k^*(n)}$*, meaning that if* $\boldsymbol{n \neq \hat{n}}$*, then* $\boldsymbol{k^*(n) < k^*(\hat{n})}$*.*

The relationship between n and k^* can be represented by a smooth curve (with n on the horizontal axis and k^* on the vertical axis) whose slope corresponds to $\partial k^*/\partial n$. Quasi-concavity has implications about the general shape of this curve:

- If $n < \hat{n}$, then $\partial k^*/\partial n > 0$;
- If $n = \hat{n}$, then $\partial k^*/\partial n = 0$;
- If $n > \hat{n}$, then $\partial k^*/\partial n < 0$.

So, to the left of $\hat{n}$ the slope is positive; to the right of $\hat{n}$ the slope is negative; and at $\hat{n}$ the curve is flat. It follows that $\hat{n}$ is the location of the highest attainable point on the curve. In other words, this stationary point, $\hat{n}$, must correspond to the maximum point of the curve.

Lemma 22-1 can be proved as follows:

Proof. Let us begin by noting that from 18-1 and 18-3 we have:

$$\mathcal{R}[k^*(n), n]\mathcal{E}[k^*(n), n] \equiv -\frac{n^2}{k^*(n)}\frac{\partial^2 k^*}{\partial n^2}.$$

Next, at a stationary point of $\boldsymbol{k^*(n)}$, we must have $\boldsymbol{\partial k^*/\partial n = 0}$, which then implies that $\boldsymbol{\mathcal{E}[k^*(n), n] = 0}$. This has a couple of implications: Firstly, from 21-2, it implies that $\boldsymbol{\mathcal{E}[v^*(n), n] = m^*(n) + p_{1|0}n}$. And, thus, $\boldsymbol{\mathcal{E}[v^*(n), n] > m^*(n)}$, given that $\boldsymbol{p_{1|0}n > 0}$. Secondly, given that $\boldsymbol{\partial k^*/\partial n = 0}$, the basic definition of an elasticity gives: $\boldsymbol{\mathcal{E}[k^*(n), n] = 0}$. Which then implies that:

$$-\frac{n^2}{k^*(n)}\frac{\partial^2 k^*}{\partial n^2} = \frac{1}{\alpha}\left\{n p_{1|1}\mathcal{E}[v^*(n), n] + \frac{1 + v^*(n)}{1 - v^*(n)}\mathcal{E}[v^*(n), n]^2 - m^*(n)^2\right\}.$$

Furthermore, since $\mathcal{E}[v^*(n), n] > m^*(n)$ and $0 < v^*(n) < 1$, it follows that:

$$\frac{1 + v^*(n)}{1 - v^*(n)}\mathcal{E}[v^*(n), n]^2 > m^*(n)^2 \;\Rightarrow\; -\frac{n^2}{k^*(n)}\frac{\partial^2 k^*}{\partial n^2} > 0$$

Since $k^*(n) > 0$, this means that $\partial^2 k^*/\partial n^2 < 0$, and so $k^*(n)$ must be concave in the vicinity of a stationary point. Hence, any stationary point must be a local maximum, and thus there cannot exist a local minimum.

Lastly, the following proof by contradiction shows that in such a situation there can only be one stationary point, and it must be the global maximum: Assume that there are two distinct stationary points, which if they were to exist must be local maxima. However, a sketch of this scenario shows that there must be a local minimum between the first local maximum and the second local maximum. (The curve must go down before it can go up again.) But this flatly contradicts what's been proved about all the stationary points being local maxima. It follows that there can only be one stationary point, and this must be a unique global maximum. This concludes the proof. ∎

22.4 THE FIRST-ORDER CONDITION

Conducting more inspections would entail higher costs (higher spending on CT, as well as a higher fraction of viable output falling prey to type-1 errors), but more inspections would also increase the detection rate, which should then boost the reliability of the economy's production processes. Moreover, the trade-off between costs and benefits suggests that there's an optimal inspection frequency.

The increase in output from having more reliable production processes will make citizens more prosperous and increase the pool of savings. Which, in turn, this yields a higher level of investment and thus a more productive economy, capable of sustaining this level of investment. Ultimately, the optimal inspection frequency will be that which sustains the highest possible equilibrium capital intensity, because it's the capital intensity that determines the prosperity of citizens.

Let $\hat{n}$ denote the value of n that maximises capita intensity in equilibrium, $k^*(n)$, where this maximum is given by $k^*(\hat{n})$. And, let $\theta(\hat{n})$, $\phi(\hat{n})$ and $\tau(\hat{n})$ denote the values of the other quantities when n is set to $\hat{n}$. Lastly, $v^*(\hat{n})$ is the value of $v^*(n)$ when n is set to $\hat{n}$ and thus $\phi(n)$ is set to $\phi(\hat{n})$.

Equations 21-2 and 20-1 imply that $\mathcal{E}[k^*(n), n] = \frac{1}{\alpha}\{\mathcal{E}[v^*(n), n] - p_{1|0}n - m^*(n)\}$, where $\mathcal{E}[v^*(n), n]$ is necessarily positive. Consequently, the sign of $\mathcal{E}[k^*(n), n]$ depends on whether $\mathcal{E}[v^*(n), n]$ is bigger or smaller than $p_{1|0}n + m^*(n)$:

$$\mathcal{E}[k^*(n), n] > 0 \quad \text{if and only if} \quad \mathcal{E}[v^*(n), n] > p_{1|0}n + m^*(n)$$

$$\mathcal{E}[k^*(n), n] = 0 \quad \text{if and only if} \quad \mathcal{E}[v^*(n), n] = p_{1|0}n + m^*(n)$$

$$\mathcal{E}[k^*(n), n] < 0 \quad \text{if and only if} \quad \mathcal{E}[v^*(n), n] < p_{1|0}n + m^*(n)$$

Moreover, Lemma 22-1 has established that $k^*(n)$ is quasi-concave with a unique stationary point, $\hat{n}$, such that $\mathcal{E}[k^*(\hat{n}), \hat{n}] = 0$. Hence, we arrive at the following first-order condition.

Proposition 22-1. *Let $\hat{n}$ denote the value of n that maximises the economy's capita intensity in equilibrium. The first-order condition for $\hat{n}$ is as follows:*

$$\mathcal{E}[v^*(\hat{n}), \hat{n}] - p_{1|0}\hat{n} - m^*(\hat{n}) = 0\,. \qquad 22\text{-}1$$

Substituting for $\mathcal{E}[v^*(\hat{n}), \hat{n}]$ in 22-2 (using equations 20-1 and 19-2) shows that $\hat{n}$ is defined by the following implicit equation:

$$m^*(\hat{n}) = \frac{p_{1|1}\hat{n}[1 - \phi(\hat{n})][1 - v^*(\hat{n})]}{\delta + g + \phi(\hat{n})} - p_{1|0}\hat{n}\,, \qquad 22\text{-}2$$

where 22-2 is composed of the following economically meaningful quantities:

- *The likelihood of type-1 errors is $p_{1|0}$ and the power of the conformance test is $p_{1|1}$.*
- *In equilibrium, the rebate rate is $[1 - \phi(\hat{n})][1 - v^*(\hat{n})]$ and the cost of conformance testing as a proportion of the economy's output is $m^*(\hat{n})$.*
- *The economy's gross investment rate is $\delta + g$ and the detection rate for defective goods is $\phi(\hat{n})$.*

The form of this first-order condition suggests that the terms represent distinct positive and negative influences on the capital intensity originating from small variations in the frequency of inspections. Each element in the expression above corresponds to a meaningful quantity, most of which would be estimable through an industry survey. From this, the optimal level of spending on conformance testing is given by the following equation:

$$\frac{\text{Cost}}{\text{GVA}} = \frac{[1 - \Pr\{\text{type 2}\}] \times (\text{Frequency}) \times (\text{Rebate Rate})}{(\text{Gross Investment Rate}) + (\text{Detection Rate})} - \Pr\{\text{type 1}\} \times (\text{Frequency})$$

This equation has several implications: Firstly, providing the RHS of the equation is positive, then 'spending' increases with 'frequency'; which accords with what is intuitively reasonable. Dividing through by 'frequency' and multiplying through by 'GVA' gives:

$$\frac{(\text{Cost})}{(\text{Frequency})} = \frac{[1 - \Pr\{\text{type 2}\}] \times (\text{Rebate Rate}) \times (\text{GVA})}{(\text{Gross Investment Rate}) + (\text{Detection Rate})} - \Pr\{\text{type 1}\} \times (\text{GVA})$$

The LHS of this equation gives us an equation for the cost of one complete sweep of the capital stock. In other words, this equation gives us the unit cost for a full round of inspections. Since this is an optimality condition, intuition suggests that the RHS must represent the net benefit from another round of inspections. The positive term representing the additional output from more reliable production processes, and the negative term being the loss due to viable output that falls prey to type-1 errors. This is explored further in the next section.

Secondly, for there to be any spending on conformance testing, it must be the case that:

$$\frac{(\text{Rebate Rate})}{(\text{Gross Investment Rate}) + (\text{Detection Rate})} > \frac{\Pr\{\text{type 1}\}}{1 - \Pr\{\text{type 2}\}}$$

Intuitively, this inequality says that for patients to keep taking their medicine, the 'symptoms' of the 'disease' needs to be worse than the side-effects of the 'treatment'. The symptoms of the disease become more extreme with a high rebate rate. But the symptoms tend to become somewhat more tolerable when there's a high 'detection rate' and/or a high gross investment rate. (If a large fraction of the capital stock is replaced each year, then there's less justification for expending more effort trying to find and fix malfunctioning machines.) Similarly, the 'treatment' becomes more attractive when it's truly an effective cure (a higher statistical power) and/or when there are fewer unfortunate side-effects (a lower likelihood of type-1 errors).

23 THE ECONOMIC INTUITION BEHIND THE OPTIMALITY CONDITION

What is the intuition behind the optimality condition? Well, the short answer is that it requires the marginal cost of another round of inspections to equal the marginal benefit. Hence, the condition has the form of a "*marginal cost must equal marginal benefit*" type of optimality condition. That is, the LHS is the marginal cost of a round of inspections and the RHS relates to the benefit from a round of inspections.

23.1 A DERIVATION OF THE OPTIMALITY CONDITION

Increasing the frequency of inspections would increase the detection rate (driving up the reliability of the production process), but it would also increase the cost of conformance testing, as well as the amount of viable output that falls prey to type-1 errors.

The reliability of production, v^*, depends on the detection rate, ϕ, which, in turn, depends on the frequency of inspections, n. It follows that an increase in the frequency of inspections will feed through to having more reliable production processes, and so there must be a positive relationship between v^* and n. Which implies that: $\partial v^*/\partial n > 0$.

Let Δn denote a small change in the frequency of inspections, and Δv^* denote the change in the reliability of production processes that comes from having more testing. Hence, Δv^* and Δn must be connected through the following identity:

$$\Delta v^* = \left(\frac{\partial v^*}{\partial n}\right)\Delta n \qquad \text{23-1}$$

The proportional increase in the reliability of production can be written as: $\Delta v^*/v^*$. From which it follows that the value of the additional output equals this quotient multiplied by the value of the economy's current output (GVA), y^*. However, an increase in the frequency of inspections will also increase the amount of viable output that falls prey to type-1 errors (false positives). If Δn is the increase in inspections, then the proportion of output falling prey to type type-1 errors rises by $p_{1|0}\Delta n$, where $p_{1|0}$ is the likelihood of making a type-1 error during a test.

The extra benefit from increased testing is the benefit from the increased reliability of the production process minus the loss from having more false positives.

$$\text{Extra Benefit} = (\Delta v^*/v^*).\,y^* - \left(p_{1|0}\Delta n\right).\,y^*$$

Using the identity above, this can be rewritten as:

$$\text{Extra Benefit} = \left(\frac{1}{v^*}\frac{\partial v^*}{\partial n} - p_{1|0}\right).y^*\Delta n \qquad 23\text{-}2$$

To find the optimal level of testing, the extra benefits need to be weighed against the extra cost. The direct cost of increased testing is the marginal cost of a round of testing multiplied by Δn. The direct cost of testing is the wage rate for engineers, ω, multiplied by the number of engineers employed doing conformance testing, e^*. So, given that n denotes the frequency of inspections, it follows that the unit cost for a round of testing is $\omega e^*/n$. If Δn is the increase in the frequency of inspections, then the extra cost from employing more engineers is given by:

$$\text{Extra Cost} = \left(\frac{\omega e^*}{n}\right).\Delta n \qquad 23\text{-}3$$

Moreover, since costs rise linearly with the frequency of testing, this is also the marginal cost of another round of testing.[39] Consequently, the optimal level of testing is reached when the extra cost equals the extra benefit from another round of testing:

$$\left(\frac{\omega e^*}{n}\right).\Delta n = \left(\frac{1}{v^*}\frac{\partial v^*}{\partial n} - p_{1|0}\right).y^*\Delta n$$

However, $\omega e^*/y^*$ gives the spending on conformance testing as a proportion of the economy's GVA, m^*. Hence, the previous expression can be rewritten as follows:

$$m^*.\left(\frac{\Delta n}{n}\right) = \left(\frac{n}{v^*}\frac{\partial v^*}{\partial n} - p_{1|0}n\right).\left(\frac{\Delta n}{n}\right)$$

This equation holds when the inspection frequency, n, is at its optimal value, $\hat{n}$. Suppose that when n is at this optimal value, the values of v^* and m^* become $v^*(\hat{n})$ and $m^*(\hat{n})$. We can cancel a factor of Δn from both sides of the previous equation to get an even neater optimality condition:

$$\frac{m^*(\hat{n})}{\hat{n}} = \left(\frac{1}{v^*}\frac{\partial v^*}{\partial n}\right)\Bigg|_{n=\hat{n}} - p_{1|0} \qquad 23\text{-}4$$

The LHS is the marginal cost of another round of conformance testing and the RHS is its marginal benefit, where both costs and benefits are expressed as proportions of the economy's GVA. Multiplying both sides through by $\hat{n}$, gives us a formula for the optimal spending on conformance testing as a proportion of the economy's GVA:

$$m^*(\hat{n}) = \left(\frac{n}{v^*}\frac{\partial v^*}{\partial n}\right)\Bigg|_{n=\hat{n}} - p_{1|0}\hat{n} \qquad 23\text{-}5$$

Note that because m^* is a monotonically increasing function of n, the optimal spending on conformance testing will correspond to the optimal inspection frequency and vice versa. In other words, the optimality condition for one is equally the optimality condition for the other.

The next step is to find an expression for $\partial v^*/\partial n$. From the chain rule of differentiation, we have the following identity:

$$\frac{\partial v^*}{\partial n} = \frac{\partial \phi}{\partial n}\frac{\partial v^*}{\partial \phi} \qquad 23\text{-}6$$

As already explained, the detection rate is given by:

$$\phi = 1 - \exp\left(-p_{1|1}n\right)$$

[39] Since $e^* = k^*/\rho$ and $a = \rho n$, we get $e^* = n/a$. A little rearrangement yields: $e^*/n = k^*/a$. Next, because a is a constant and $\partial k^*/\partial n|_{n=\bar{n}} = 0$, it follows that e^*/n does not change as n varies at least close to $n = \bar{n}$.

where, $p_{1|1}$ is the power of the test, and n is the frequency of inspections. Whilst the power of the test is fixed by technology, the frequency of inspections is under society's control, and there is a positive relationship between the frequency of inspections and the detection rate:

$$\frac{\partial \phi}{\partial n} = p_{1|1} \exp(-p_{1|1} n) = p_{1|1}(1 - \phi)$$

Hence, our expression for $\partial v^*/\partial n$ can be rewritten as:

$$\frac{\partial v^*}{\partial n} = p_{1|1}(1 - \phi)\frac{\partial v^*}{\partial \phi}$$

In other words, because v^* depends on ϕ, a change in frequency of inspections, n, will feed through to a change in the reliability of production.

The next step is to find an expression for $\partial v^*/\partial \phi$, which can be done by revisiting the equilibrium condition for $\dot{v}(t) = 0$, and working through the consequences of a change in the detection rate. We can think of the system involving flows of probability in and out of two buckets: one for 'good' machines and another for machines that have gone 'bad'. For the system to be in equilibrium, the inflow to $1 - v^*$ needs to outflow from $1 - v^*$. (The same goes for v^*.) From this, it can be shown that the equilibrium condition for $\dot{v}(t) = 0$ is as follows: $(1 - v^*)(\delta + g + \phi) = \varepsilon v^*$. Furthermore, as previously shown, this leads to an equilibrium in which the reliability of production is given by:

$$v^* = \frac{\delta + g + \phi}{\delta + g + \phi + \varepsilon}$$

For future reference, note that this means: $v^*(\delta + g + \phi + \varepsilon) = \delta + g + \phi$.

Next, we consider the effect of a change in the detection rate on the reliability of the production process. Differentiating the equilibrium condition yields the following result:

$$\frac{\partial v^*}{\partial \phi} = \frac{(1 - v^*)}{\delta + g + \phi + \varepsilon} \qquad \text{23-7}$$

This shows that increasing the detection rate will always increase the reliability of production. Substituting this result for $\partial v^*/\partial \phi$ into our previous equation for $\partial v^*/\partial n$ gives us the following expression:

$$\frac{\partial v^*}{\partial n} = \frac{p_{1|1}(1 - \phi)(1 - v^*)}{\delta + g + \phi + \varepsilon} \qquad \text{23-8}$$

Suppose that when n is at its optimal value, the detection rate becomes $\phi(\hat{n})$. Evaluating this derivative at the optimal inspection frequency gives:

$$\left(\frac{\partial v^*}{\partial n}\right)\bigg|_{n=\hat{n}} = \frac{p_{1|1}[1 - \phi(\hat{n})][1 - v^*(\hat{n})]}{\delta + g + \phi(\hat{n}) + \varepsilon} \qquad \text{23-9}$$

Substituting this result into our earlier optimality condition gives:

$$m^*(\hat{n}) = \frac{\{p_{1|1}\hat{n}\}.[1 - \phi(\hat{n})][1 - v^*(\hat{n})]}{v^*(\hat{n})[\delta + g + \phi(\hat{n}) + \varepsilon]} - \{p_{1|0}\hat{n}\}$$

From the equilibrium condition for $\dot{v}(t) = 0$, we get $v^*(\delta + g + \phi + \varepsilon) = \delta + g + \phi$, and so the expression for $m^*(\hat{n})$ becomes:

$$m^*(\hat{n}) = \frac{\{p_{1|1}\hat{n}\}.[1 - \phi(\hat{n})][1 - v^*(\hat{n})]}{\delta + g + \phi(\hat{n})} - \{p_{1|0}\hat{n}\} \qquad \text{23-10}$$

Lastly, since $[1 - \phi(\hat{n})][1 - v^*(\hat{n})]$ is the rebate rate, $\Omega^*(\hat{n})$, the condition for the optimal amount of testing can be rewritten as:

$$m^*(\hat{n}) = \frac{\{p_{1|1}\hat{n}\}.\Omega^*(\hat{n})}{\delta + g + \phi(\hat{n})} - \{p_{1|0}\hat{n}\}$$

23.2 A GRAPHICAL REPRESENTATION OF THE RELATIONSHIP

For us to better understand the optimality condition, it's helpful to have a graphical representation of how the marginal cost and benefit each respond to variations in the frequency of inspections. Let us start by recalling that that we can write the first-order condition as:

$$\frac{m^*}{\hat{n}} = \frac{p_{1|1}(1-\phi)(1-v^*)}{\delta + g + \phi} - p_{1|0}$$

As discussed, this equation has the form of a "marginal benefits equal marginal cost" kind of optimality condition: The LHS is the marginal cost and the RHS is the marginal benefit. Also, for future reference, notice that the RHS of this expression will decrease if $p_{1|0}$ increases. In other words, the marginal benefit from another round of inspections decreases if the likelihood of type-1 errors increases.

Let us first consider the LHS of the equation (the marginal cost of an inspection). From equation 15-15 we have $m^* = \tau k^*/y^*$, which means the LHS can be rewritten as:

$$\frac{m^*}{n} = \frac{\tau k^*}{n y^*}$$

Secondly, from equation 10-7 we have $\tau = (\omega/a)n$, which then implies that:

$$\frac{m^*}{n} = \frac{\omega}{a} \times \frac{k^*}{y^*}$$

An engineer's wage rate, ω, and span-of-control, a, are basic parameters of the model, and so ω/a is a fixed constant. Hence, the cost of another round of inspections must be proportional to the equilibrium capital ratio. Lastly, from equation 17-18, we have the following expression for the equilibrium capital ratio:

$$\frac{k^*}{y^*} = \frac{\beta}{r^*}$$

This shows that the capital ratio is inversely related to the rental rate, r^*. From which it follows that the cost of another round of inspections is given by:

$$\frac{m^*}{n} = \frac{\omega\beta}{a r^*}$$

Notice that because r^* is essentially a fixed constant of the system, m^*/n must also be a constant. In other words, the marginal cost of a round of inspections is essentially another fixed parameter.

Next we turn to the RHS of the optimality condition (the marginal benefit of another round of inspections.) How will the RHS respond to increasing or decreasing the frequency of inspections? Since $p_{1|0}$ is a parameter that doesn't depend in any way on n, we focus in on the first term, which correspond to the gross benefit from another round of inspections:

$$\text{gross benefit} = \frac{p_{1|1}(1-\phi)(1-v^*)}{\delta + g + \phi}$$

How will the numerator and denominator of this expression respond to a change in n? The answer can be found by considering the corresponding changes in ϕ and v^*. So, from equations 13-6 and 17-11, we have:

$$\frac{\partial \phi}{\partial n} = \wp_{1|1}(1 - \phi) > 0$$

$$\frac{\partial v^*}{\partial n} = \frac{\wp_{1|1}(1 - \phi)(1 - v^*)}{\delta + g + \phi + \varepsilon} > 0$$

Hence, ϕ and v^* are both increasing functions of n. Which then implies that $(1 - \phi)(1 - v^*)$ decreases as n increases; and $\delta + g + \phi$ increases as n increases. Consequently, the gross benefit from another round of inspections will decrease as n increases. (Because the numerator decreases whilst the denominator increases.) That is, there's always diminishing returns from increasing the frequency of inspections.

So, to summarise: the cost of another round of inspections is, essentially, constant, whereas the marginal benefit from another round of inspections decreases as the frequency of inspections increases. These findings are brought together in the figure below:

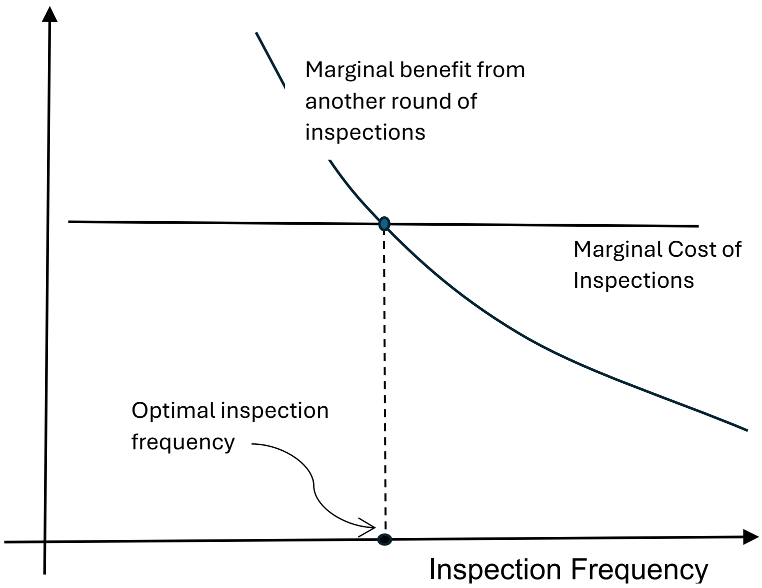

Figure 5: Marginal cost and benefit of another round of inspections

The horizontal line represents the fixed marginal cost of each round of inspections (the LHS of the optimality condition). The downward sloping curve represents the marginal benefit from another round of inspections (the RHS of the optimality condition). The point where these intersect determines the optimal inspection frequency: the point where marginal cost equals the marginal benefit.

Lastly, this diagram helps us to understand what would happen to the optimal frequency of inspections if the likelihood of type-1 errors was to increase. As already discussed, if $\wp_{1|0}$ increases, then the expression on the RHS of the optimality condition must decrease. This corresponds to a downwards shift in the curve representing the marginal benefit from another round of inspections. Since m^*/n is essentially a fixed parameter, that doesn't depend in on $\wp_{1|0}$, the marginal cost of another round of inspections is basically unchanged; and so, there is no movement in the horizontal line. Consequently, an increase in the likelihood of type-1 errors must lead to a decrease in the optimal frequency of inspections.

24 ESTIMATING THE MODEL'S PARAMETERS

This section uses a numerical analysis to find the values of the unobservable parameters. The aim is to find values that are consistent with the formulae for the equilibrium values of the variables and the values of the known parameters.

Some of the parameters in our model are difficult to determine from macroeconomic data. However, the scrap rate, rebate rate, and spending on conformance testing are observable quantities, as is an engineer's "portion size". How can we use these values to determine the remaining unknowns in the model?

Firstly, it's assumed that the system has settled into what we call the "best attainable equilibrium". Secondly, the analysis above yielded the following formulae:

- Expressions for the detection rate and the regret rate.
- Expressions for the success rate and rebate rate.
- An expression for the "pace of testing" in terms of the "portion size" and the frequency of inspections.
- An expression for the reliability of production process in the system's equilibrium.
- A first-order condition for the optimal frequency of inspections.

So, if the system is in this "best attainable equilibrium", then these equations can be solved to determine the unknowns. Specifically, we can derive an implicit equation for the detection rate, ϕ, and solving this equation then makes it possible to infer the values of the other unknowns.

The central issue is that the value of the portion-size, ρ, is known, whereas, the inspection frequency, n, is an unknown parameter. However, recall that $n = a/\rho$, where a is the pace of testing and ρ is the portion size. Thus, if $\hat{n}$ is the optimal frequency, then the optimal portion size is $\rho(\hat{n}) = a/\hat{n}$. In other words, given that the pace of testing, a, is a completely fixed parameter, there is a one-to-one relationship between ρ and $\hat{n}$. It follows that the first-order condition for the optimal frequency is also a first-order condition for the optional portion-size.

Definition: *The system is said to be in the "best attainable equilibrium" when the following two conditions are satisfied: Firstly, the reliability of production processes, v, and capital intensity, k, are at their steady state values: v^* and k^*. Secondly, the frequency of inspections, n, is at the level which maximises the steady state value of citizens' per capita income. Hence, the necessary and sufficient conditions for the system to be in the "best attainable equilibrium" are:*

(a) *Both v and k are at their steady state values, which implies that $v \to v^*$, $k \to k^*$.*
(b) *The frequency of inspections, $\hat{n}$, satisfies the first-order condition for optimality given by equation 22-1.*

24.1 SETTING UP THE PROBLEM

This section sets up the mathematical problem to be solved. It lists the equations and the unknowns. (For the problem to be solvable, there should be as many equations as unknowns.)

There are seven unknowns:

1. The optimal frequency of inspections, $\hat{n}$.
2. The pace of inspections, a.
3. The transition rate, ε.
4. The reliability of the production process in the system's equilibrium, $v^*(\hat{n})$.
5. The detection rate, $\phi(\hat{n})$.
6. The regret rate, $\theta(\hat{n})$.
7. The likelihood of false positives, $p_{1|0}$.

There are seven equations:

1. The detection rate is $\phi(\hat{n}) = 1 - \exp(-p_{1|1}\hat{n})$, where $p_{1|1} = 99.9\%$.
2. The regret rate is $\theta(\hat{n}) = 1 - \exp(-p_{1|0}\hat{n})$.
3. The success rate is $\Lambda^*(\hat{n}) = [1 - \theta(\hat{n})]v^*(\hat{n})$, where $\Lambda^*(\hat{n}) = 98.4\%$.
4. The rebate rate is $\Omega^*(\hat{n}) = [1 - \phi(\hat{n})][1 - v^*(\hat{n})]$, where $\Omega^*(\hat{n}) = 1.5\%$.
5. The pace of testing is $a = \hat{n}\rho(\hat{n})$, where the "portion size" is $\rho(\hat{n}) = £5.1$ million.
6. The reliability of production processes in equilibrium is given by:

$$v^*(\hat{n}) = \frac{\delta + g + \phi(\hat{n})}{\delta + g + \phi(\hat{n}) + \varepsilon},$$

   where the gross investment rate is $\delta + g = 6.3\%$.

7. The optimal amount spent on conformance testing is given by:

$$m^*(\hat{n}) = \frac{\{p_{1|1}\hat{n}\}.\,\Omega^*(\hat{n})}{\delta + g + \phi(\hat{n})} - \{p_{1|0}\hat{n}\},$$

   where $m^*(\hat{n}) = 2\%$ is the amount spent on testing as a percentage of output; the rebate rate is $\Omega^*(\hat{n}) = 1.5\%$; the statistical power of a test is $p_{1|1} = 99.9\%$; and the gross investment rate is $\delta + g = 6.3\%$.

Proposition 24-1: We have a system of simultaneous equations with seven unknowns and seven independent equations. Hence, this system has a unique solution, meaning that we can solve these equations to determine the unknowns.

24.2 AN IMPLICIT EQUATION FOR THE DETECTION RATE

The optimality condition can be used to generate an implicit equation for the detection rate, which can then be solved numerically.

The condition for the optimal spending on conformance testing can be written as:

$$m^*(\hat{n}) = \frac{\{p_{1|1}\hat{n}\}.\,\Omega^*(\hat{n})}{\delta + g + \phi(\hat{n})} - \{p_{1|0}\hat{n}\}$$

Furthermore, as already discussed, we have estimates for many of the variables and parameters in this expression: $\delta + g = 6.3\%$; $m^*(\hat{n}) = 2\%$; and $\Omega^*(\hat{n}) = 1.5\%$.

The next step is to express the elements in curly brackets ($p_{1|1}\hat{n}$ and $p_{1|0}\hat{n}$) in terms of the detection rate, $\phi(\hat{n})$: Since $\phi(\hat{n}) = 1 - \exp\,(-p_{1|1}\hat{n})$ and $\theta(\hat{n}) = 1 - \exp\,(-p_{1|0}\hat{n})$, we have:

$$p_{1|1}\hat{n} = \ln\left[\frac{1}{1 - \phi(\hat{n})}\right]$$

$$p_{1|0}\hat{n} = \ln\left[\frac{1}{1 - \theta(\hat{n})}\right]$$

Hence, we can rewrite the optimality condition as:

$$m^*(\hat{n}) = \ln\left[\frac{1}{1 - \phi(\hat{n})}\right]\left[\frac{\Omega^*(\hat{n})}{\delta + g + \phi(\hat{n})}\right] - \ln\left[\frac{1}{1 - \theta(\hat{n})}\right]$$

Furthermore, the success rate is $\Lambda^*(\hat{n}) = [1 - \theta(\hat{n})]v^*(\hat{n})$, which implies that:

$$1 - \theta(\hat{n}) = \frac{\Lambda^*(\hat{n})}{v^*(\hat{n})}$$

Whilst the rebate rate is $\Omega^*(\hat{n}) = [1 - \phi(\hat{n})][1 - v^*(\hat{n})]$, which implies that:

$$v^*(\hat{n}) = 1 - \frac{\Omega^*(\hat{n})}{1 - \phi(\hat{n})}$$

So, the previous expression for $1 - \theta(\hat{n})$ becomes:

$$1 - \theta(\hat{n}) = \frac{\Lambda^*(\hat{n})}{1 - \{\,\Omega^*(\hat{n})/[1 - \phi(\hat{n})]\,\}}$$

Which after a little rearrangement this yields:

$$\frac{1}{1 - \theta(\hat{n})} = \frac{[1 - \phi(\hat{n})] - \Omega^*(\hat{n})}{\Lambda^*(\hat{n})[1 - \phi(\hat{n})]}$$

Using this result, the optimality condition can be rewritten as:

$$m^*(\hat{n}) = \left[\frac{\Omega^*(\hat{n})}{\delta + g + \phi(\hat{n})}\right] \ln\left[\frac{1}{1 - \phi(\hat{n})}\right] - \ln\left[\frac{1 - \phi(\hat{n}) - \Omega^*(\hat{n})}{\Lambda^*(\hat{n}) - \Lambda^*(\hat{n})\phi(\hat{n})}\right] \qquad \text{24-1}$$

Notice that $\phi(\hat{n})$ is now the only unknown in this equation, making it an implicit equation for the detection rate. So, the next step is to find a value of $\phi(\hat{n})$ that yields a value of $m^*(\hat{n})$ that's consistent with the known spending on conformance testing. Hence, the best estimate of $\phi(\hat{n})$ is that which makes $m^*(\hat{n})$ as close as possible to the observed value of 2%.

24.3 FINDING THE SOLUTION

To solve the equation given above, it's easier to work with $x \equiv 1 - \phi(\hat{n})$. In terms of x, the equation becomes:

$$m^*(\hat{n}) = \frac{\Omega^*(\hat{n}).\ln(1/x)}{\delta + g + 1 - x} - \ln\left[\frac{x - \Omega^*(\hat{n})}{\Lambda^*(\hat{n}).x}\right]$$

Some useful estimates for the other variables and parameters in the equation are as follows: $\delta + g = 6.3\%$; $m^*(\hat{n}) = 2\%$; $\Lambda^*(\hat{n}) = 94.8\%$; and $\Omega^*(\hat{n}) = 1.5\%$. Substituting in these estimates into the formula above gives:

$$2\% = \frac{1.5\% \times \ln(1/x)}{6.3\% + 1 - x} - \ln\left(\frac{x - 1.5\%}{94.8\% \times x}\right) \qquad \text{24-2}$$

Since x is the only unknown, the equation can be solved using graphical and/or numerical methods. This implicit equation for x can be rewritten as $\Theta(x) = 0$, where $\Theta(\cdot)$ is defined as follows:

$$\Theta(x) := \frac{1.5\% \times \ln(1/x)}{6.3\% + 1 - x} - \ln\left(\frac{x - 1.5\%}{94.8\% \times x}\right) - 2\%. \qquad \text{24-3}$$

The next step is to find the solution to this equation using graphical and numerical methods. A graphical method involves a graph in which $\Theta(x)$ is plotted on the vertical axis and x is plotted on the horizontal axis. The solution corresponds to the point where the curve crosses the horizontal axis. This shows that there's a solution between 0.30 and 0.35.

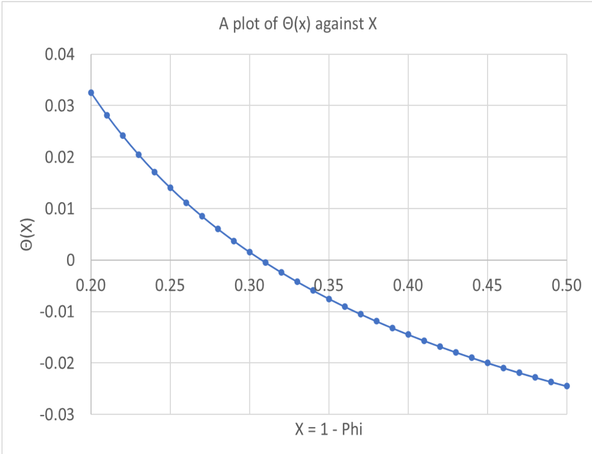

Figure 6: A graphical solution for the implicit equation

A more precise estimate can be arrived at using a numerical method. Newton's method is an iterative approach based on the following recurrence formula:

$$x_{i+1} = x_i - \frac{\Theta(x_i)}{\Theta'(x_i)} \qquad \text{24-4}$$

Where the derivative of $\Theta(x)$ is given by:

$$\Theta'(x) = \frac{1.5\% \times \ln(1/x)}{(6.3\% + 1 - x)^2} - \frac{1.5\%}{x \times (6.3\% + 1 - x)} - \frac{1.5\%}{x \times (x - 1.5\%)} \qquad \text{24-5}$$

Starting from $x_1 = 30\%$, we proceed through the iterations in the table below:

Table 4: A numerical solution to the implicit equation

i	x	Θ(x)	Θ'(x)
1	0.3000000000000	0.0015307271695	-0.2099435578669
2	0.3072911366512	0.0000344667337	-0.2006012021305
3	0.3074629538355	0.0000000182332	-0.2003890216296
4	0.3074630448246	0.0000000000000	-0.2003889093594
5	0.3074630448247	0.0000000000000	-0.2003889093594
6	0.3074630448247		

The values of x_i quickly converge to a determinate value; and, after six iterations, the results are the same to high number of decimal places. We arrive at: $x = 30.7\%$, accurate to 3 significant figures.

24.4 WORKING BACK TO FIND THE UNKNOWNS

Numerical analysis showed that $x = 30.7\%$, which then implies that:

$$\phi(\hat{n}) = 69.3\%.$$

This means that the probability of detecting defective outputs before they reach the market is almost 70%. This result will now be used to generate values for the remaining unknowns in the model.

The formula for the detection rate can be rearranged to give:

$$\hat{n} = \frac{1}{\wp_{1|1}} \ln\left[\frac{1}{1 - \phi(\hat{n})}\right] \qquad \text{24-6}$$

Since $\phi(\hat{n}) = 69.3\%$ and $p_{1|1} = 99.9\%$, this yields:

$$\hat{n} = \frac{1}{99.9\%} \ln\left(\frac{1}{1 - 69.3\%}\right) = 1.18$$

This means that, on average, a portion of machinery will receive 1.18 inspections per year. So, this amounts to an inspection every 44 weeks.

The formula for the pace of inspections says that $a = \hat{n} \times \rho(\hat{n})$. Since $\hat{n} = 1.18$ and $\rho(\hat{n}) =$ £5.1 million, this yields:

$$a = 1.18 \times £5.1\ \text{million} = £6.0\ \text{million}$$

Hence, an engineer can inspect £6 million worth of capital equipment each year, which gives an estimate of the amount of capital they can reasonably be expected to supervise (i.e., their span of control).

The formula for the rebate rate can be rearranged to give us an expression for the reliability of production processes:

$$v^*(\hat{n}) = 1 - \frac{\Omega^*(\hat{n})}{1 - \phi(\hat{n})} \qquad \text{24-7}$$

Since $\phi(\hat{n}) = 69.3\%$ and $\Omega^*(\hat{n}) = 1.5\%$, this yields:

$$v^*(\hat{n}) = 1 - \frac{1.5\%}{1 - 69.3\%} = 95.1\%$$

This means that less than 5% of the output is defective, but much of this will be picked up by conformance tests before it enters the market.

The formula for v^* can be rearranged to give a formula for the transition rate:

$$\varepsilon = \frac{[1 - v^*(\hat{n})][\delta + g + \phi(\hat{n})]}{v^*(\hat{n})} \qquad \text{24-8}$$

Since $\delta + g = 6.3\%$, $\phi(\hat{n}) = 69.3\%$, and $v^*(\hat{n}) = 95.1\%$, this yields:

$$\varepsilon = \frac{(1 - 95.1\%) \times (6.3\% + 69.3\%)}{95.1\%} = 3.9\%$$

So, each year, 3.9% of the “good” machinery in the capital stock will begin to malfunction, and so start producing defective outputs. This is an estimate of the intrinsic instability in the economy’s production processes (i.e., transition rate) that when they flip from “good” to “bad” leads to malfunctions in production.

The formula for the success rate can be rearranged to give us an expression for the regret rate:

$$\theta(\hat{n}) = 1 - \frac{\Lambda^*(\hat{n})}{v^*(\hat{n})} \qquad \text{24-9}$$

Since $v^*(\hat{n}) = 95.1\%$ and $\Lambda^*(\hat{n}) = 94.8\%$, this yields:

$$\theta(\hat{n}) = 1 - \frac{94.8\%}{95.1\%} = 0.3\%$$

This means that only 0.3% of “good” output falls prey to type-1 errors in the testing process.

Lastly, the formula for the regret rate can be rearranged to give an expression for the likelihood of type-1 errors in the testing process:

$$p_{1|0} = \frac{1}{\hat{n}} \ln\left[\frac{1}{1 - \theta(\hat{n})}\right] \qquad \text{24-10}$$

Since $\hat{n} = 1.18$ and $\theta(\hat{n}) = 0.3\%$, this yields:

$$p_{1|0} = \frac{1}{1.18}\ln\left(\frac{1}{1 - 0.3\%}\right) = 0.3\%$$

Hence, there's a 0.3% chance of a test giving a "false positive" when it is applied to "good" outputs. Notice that because $\theta(\hat{n})$ is so small, a Taylor expansion of the expression above gives: $p_{1|0} \approx \theta(\hat{n})/\hat{n}$.

Proposition 24-2: *Consider the UK's economy during the five-year period from 2015 to 2019. Suppose that during this period the economy has settled into its best attainable equilibrium. From this, it can then be shown that the system of simultaneous equations yields the following results:*

(a) Consistent values for the model's "unknown" parameters are:
- *The transition rate is* $\varepsilon = 3.9\%$.
- *The likelihood of a type-1 error is* $p_{1|0} = 0.3\%$.
- *The pace of testing is* $a = £6.0$ *million.*

(b) Consistent values for the economy's "unknown" quantities are:
- *The frequency of inspections is* $\hat{n} = 1.18$.
- *The regret rate is* $\theta(\hat{n}) = 0.3\%$.
- *The detection rate is* $\phi(\hat{n}) = 69.4\%$.

(c) In equilibrium, the reliability of production processes is $v^*(\hat{n}) = 95.1\%$.

24.5 COMPARATIVE STATICS (AGAIN)

Using these results it's possible to estimate the following elasticities and curvatures: Equations 19-1 and 19-2 supply expressions for the elasticities of θ and ϕ with respect to $\hat{n}$:

$$\mathcal{E}(\theta, \hat{n}) = \hat{n} p_{1|0} (1 - \theta)/\theta \qquad 24\text{-}11$$

$$\mathcal{E}(\phi, \hat{n}) = \hat{n} p_{1|1} (1 - \phi)/\phi \qquad 24\text{-}12$$

Using the values listed in Proposition 24-1 and Proposition 24-2, these expressions can be evaluated as follows:

$$\mathcal{E}(\theta, \hat{n}) = 1.18 \times 0.3\% \times 99.7\%/0.3\% = 117.6\%$$

$$\mathcal{E}(\phi, \hat{n}) = 1.18 \times 99.9\% \times 30.6\%/69.4\% = 52.0\%$$

So, although, θ and ϕ increase with $\hat{n}$, the effect on θ is large compared to that on ϕ. This implies that increasing the frequency of inspections beyond $\hat{n}$ would lead to a large loss from more false positives and a relatively small benefit of detecting a few more of the malfunctions in production.

Equation 20-1 supplies an expression for the elasticity of v^* with respect to $\hat{n}$:

$$\mathcal{E}(v^*, \hat{n}) = \frac{\phi \mathcal{E}(\phi, \hat{n})(1 - v^*)}{\delta + g + \phi} \qquad 24\text{-}13$$

Using the values in Proposition 24-1 and Proposition 24-2, along with our earlier estimate of $\mathcal{E}(\phi, \hat{n})$, gives:

$$\mathcal{E}(v^*, \hat{n}) = \frac{69.4\% \times 52.0\% \times 4.9\%}{6.3\% + 69.4\%} = 2.3\%$$

Note the very small size of this elasticity. Thus, although, there's a positive relationship, this implies that v^* responds inelastically to an increase in $\hat{n}$. Hence, the reliability of production, v^*, won't improve much even if the frequency of inspections, $\hat{n}$, was to double.

Lastly, from equation 20-2, we have the following expression for the curvature:

$$\mathcal{R}(v^*, \hat{n}) = \frac{\phi}{1-\phi}\mathcal{E}(\phi, \hat{n}) + \frac{2v^*}{1-v^*}\mathcal{E}(v^*, \hat{n}) \qquad 24\text{-}14$$

Using the values in Proposition 24-1 and Proposition 24-2, along with estimates of $\mathcal{E}(\phi, \hat{n})$ and $\mathcal{E}(v^*, \hat{n})$, gives:

$$\mathcal{R}(v^*, \hat{n}) = \frac{69.4\%}{30.6\%} \times 52.0\% + \frac{2 \times 95.1\%}{4.9\%} \times 2.3\% = 207.2\%$$

This implies that the curve created by plotting the reliability of production, v^*, as a function of inspection frequency, n, has a relatively high curvature at the optimal impaction frequency, $\hat{n}$.

This concludes Part 3 of this report. The next step is to explore the effect of calibration-related measurement uncertainty on the Relative Standard Deviation (RDS) of measurements used for conformance testing.

25 MEASUREMENT UNCERTAINTY AND THE LIKELIHOOD OF TYPE-1 ERRORS

This section begins Part 4 of this report, which introduces the concepts of "basic calibration" and "precise calibration". Subsequent sections explore what would happen if the UK were to take traceability from a foreign NMI, such as, VSL in the Netherlands. However, the next step is to develop the link between measurement uncertainty and the likelihood of type-1 errors in the testing process.

This section builds on an earlier report that gave a model for the contribution of the calibration labs to the effectiveness of conformance testing activities.[40] This contribution happens through reducing (or eliminating) systematic errors in the readings coming from the measuring instruments, and so improves the reliability of conformance tests.

25.1 MEASUREMENT UNCERTAINTY IN THE STEADY STATE

As already discussed, it's possible to derive a formula connecting the relative standard deviation (RDS) of the tests, σ, to the likelihood of type-1 errors, $p_{1|0}$, and the statistical power of the test, $p_{1|1}$:

$$\sigma = \frac{1}{\Phi^{-1}(1 - p_{1|0}) - \Phi^{-1}(1 - p_{1|1})} \qquad 25\text{-}1$$

This formula implies that, for a given a value of σ, there's a tension between the statistical power of the test and the likelihood of type-1 errors. The value of σ can be found by substituting values for $p_{1|0}$ and $p_{1|1}$ into this formula. From the results of the previous section, we have $p_{1|0} = 0.3\%$ and $p_{1|1} = 99.9\%$, which then gives:

$$\sigma = \frac{1}{\Phi^{-1}(99.7\%) - \Phi^{-1}(0.1\%)}$$

Evaluating the terms in this expression yields:

$$\sigma = \frac{1}{2.75 - (-3.09)} = 17.1\%$$

[40] King, M. & Nayak, S. (2023). An Economic Model for the Value Attributable to High-Quality Calibrations by Reducing Mistakes in Conformance Testing. NPL Report. IEA 19.

So, in the steady state, the relative standard deviation (RSD) amounts to about 17% of the required tolerance.

25.2 UNCERTAINTY INCREASES THE LIKELIHOOD OF TYPE-1 ERRORS

Measurement errors mean that repeated measurements of a single sample tend to vary. The relative standard deviation (RSD), σ, represents potential variation in the measured values for a single sample as a proportion of the required tolerance. This section begins by finding an expression connecting changes in σ to changes in the likelihood of type-1 and type-2 errors, which are denoted by $\wp_{1|0}$ and $\wp_{0|1}$, respectively. It is helpful to begin by defining the PDF for measurement errors in the testing process.

Definition: *From the fundamentals of probability theory, the first derivative of a CDF gives the corresponding PDF (Probability Density Function):*

$$\Phi'(z) := \frac{d\Phi}{dz} \tag{25-2}$$

Hence, this PDF can be written as follows:

$$\Phi'(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}z^2\right) \tag{25-3}$$

It will be assumed that the power of the test isn't something that's negotiable (so that the likelihood of type-2 errors is held constant), meaning that buyers won't tolerate less rigorous tests that would permit more defective output to enter the market. Overseas buyers would be particularly resistant because it's more difficult for them to return any defective goods they receive.

Next, imagine that something was to happen to the measurement infrastructure such that there was an increase in the relative standard deviation (RSD) of the measurement process. Because there's no change in the likelihood of type-2 errors, a change in the RSD leads to a change in the likelihood of type-1 errors:

Proposition 25-1: $\boldsymbol{\sigma}$ *denotes the relative standard deviation (RSD) of the measurement process and its variance is* $\boldsymbol{\sigma^2}$*. The likelihood of type-1 error (false positive) is* $\boldsymbol{\wp_{1|0}}$ *and the statistical power of the test is* $\boldsymbol{\wp_{1|1}}$*. Lastly, as already discussed, the reciprocal of the relative standard error (RSD) is connected to the likelihoods of these errors through the following formula:* $\mathbf{1/\boldsymbol{\sigma} = \boldsymbol{\Phi}^{-1}(1-\boldsymbol{p}) - \boldsymbol{\Phi}^{-1}(1-\boldsymbol{q})}$*, where* $(\boldsymbol{p},\boldsymbol{q}) \equiv (\boldsymbol{\wp_{1|0}},\boldsymbol{\wp_{1|1}})$ *and* $\boldsymbol{\Phi}(\cdot)$ *is the CDF of the standard normal distribution. From this formula it can be shown that:*

(a) When $\boldsymbol{\sigma}$ *changes, the resulting change in* $\boldsymbol{\wp_{1|0}}$ *and* $\boldsymbol{\wp_{1|1}}$ *are connected to the variance of the measurement process through the following formula:*

$$\frac{1}{\sigma^2} = \frac{dp/d\sigma}{\Phi'\left(\Phi^{-1}(1-p)\right)} - \frac{dq/d\sigma}{\Phi'\left(\Phi^{-1}(1-q)\right)} \tag{25-4}$$

where $(p,q) \equiv (\wp_{1|0},\wp_{1|1})$ *are the likelihoods of the type-1 and the statistical power of the test;* $\Phi'(\cdot)$ *is the first derivative of* $\Phi(\cdot)$*; and* $\Phi^{-1}(\cdot)$ *is the inverse of the* $\Phi(\cdot)$*.*

(b) In principle, a change in the relative standard error (RSD) could alter the values of both $\boldsymbol{\wp_{1|0}}$ *and* $\boldsymbol{\wp_{1|1}}$*. But, since* $\boldsymbol{\wp_{1|1}}$ *is assumed to be held fixed, the change in* $\boldsymbol{\wp_{1|0}}$ *is given by the following formula:*

$$\Delta\boldsymbol{\wp_{1|0}} = \frac{\Delta\sigma}{\sigma}\Phi'\left(\Phi^{-1}(1-\boldsymbol{\wp_{1|0}})\right)\left[\boldsymbol{\Phi}^{-1}(\mathbf{1}-\wp_{1|0}) - \boldsymbol{\Phi}^{-1}(1-\wp_{1|1})\right], \tag{25-5}$$

where $\Delta\sigma$ *denotes the change in the relative standard deviation (RSD).*

Proof. To simply the notation, let $p \equiv \wp_{1|0}$ and $q \equiv \wp_{1|1}$. Taking the reciprocal of our expression for σ gives:

$$\frac{1}{\sigma} = \Phi^{-1}(1-p) - \Phi^{-1}(1-q)$$

The total derivative of $1/\sigma$ is as follows:

$$-\frac{1}{\sigma^2} = \frac{dp}{d\sigma}\frac{\partial}{\partial p}[\Phi^{-1}(1-p)] - \frac{dq}{d\sigma}\frac{\partial}{\partial q}[\Phi^{-1}(1-q)]$$

Next, for a function $\Phi(\cdot)$ with an inverse $\Phi^{-1}(\cdot)$, the derivative of this inverse is given by:

$$\frac{d\Phi^{-1}}{dz} = \frac{1}{\Phi'\left(\Phi^{-1}(z)\right)}.$$

Where this useful identity is based on the "inverse function rule" of calculus. Hence, we arrive at equation 25-4, which completes the proof of part (a). Next, it's assumed that there's no change in the likelihood of type-2 errors, even in circumstances where σ increases, which implies that $dq/d\sigma = 0$. Consequently, equation 25-4 implies that a change in σ leads to a formula for $dp/d\sigma$:

$$\frac{dp}{d\sigma} = \frac{1}{\sigma^2}\Phi'\left(\Phi^{-1}(1-p)\right).$$

Moreover, by letting $\Delta\sigma$ and Δp denote changes in the RSD and the likelihood of type-1 errors, this can be approximated as follow:

$$\frac{\Delta p}{\Delta\sigma} \approx \frac{1}{\sigma^2}\Phi'\left(\Phi^{-1}(1-p)\right)$$

A little further rearrangement completes the proof of part (b); where we substituted for $1/\sigma$ using our initial formula. ■

Notice that equation 25-5 provides a convenient way of converting a change in σ into a change in $\wp_{1|0}$. That is, it shows how to convert an increase in the relative standard deviation (RDS) of the measurement process into an increase in the likelihood of type-1 errors (false-positives).

25.3 "BASIC CALIBRATION" AND "PRECISE CALIBRATION"

To appreciate the role of calibration, it's helpful to imagine many supposed 1m rulers, none of which are exactly 1m long: Some are a little longer and some are a little shorter, and the individual deviations from 1m are systematic errors. The owner of a specific ruler doesn't know whether their ruler is a little longer or shorter than 1m, but they can know the distribution of the systematic errors.

On their website[41], Duncan Aviation Calibration Services explain the concept of a Test Accuracy Ratio (TAR) as follows:

"TAR is the comparison between the accuracy of a tool (Unit Under Test or UUT) and the reference standard used to calibrate it. Metrology labs aim for a minimum TAR of 4:1, meaning the standard should be four times more accurate than the tool."

[41] An acceptable Test Accuracy Ratio (TAR) is 4:1. A useful reference for this is: https://www.duncanaviation.aero/intelligence/2019/January/aircraft-tool-calibration-what-is-test-accuracy-ratio

Generally, part of the uncertainty associated with a measurement comes from systematic errors, but such errors can be greatly reduced, or even eliminated, through very accurate calibration. Hence, we suppose that there are two grades of calibration:

- “Precise calibration” can all but eliminate the calibration-related component of uncertainty. (“High quality” calibration will be used interchangeably with “precise” calibration.)
- “Basic calibration” can reduce the calibration-related component of uncertainty but does not eliminate it. Suppose that, in this situation, the calibration-related component of uncertainty can be as much as 25% of the total uncertainty.

Regular access to precisely calibrated instruments depends on businesses having easy access to a large network of accredited calibration labs that are widely distributed across the country. These labs supply “precise calibrations” that are traceable to the primary standards maintained by the NMS laboratories.

26 EFFECT ON THE UNCERTAINTY OF USING “BASIC CALIBRATIONS”

Without the NMS labs, the UK would need to rely on a foreign National Measurement Institute (NMI) to ensure the traceability of its calibrations. In this situation, it would be much more difficult and expensive for UK-based businesses to access precise calibrations. Hence, “basic calibrations” would remain available even without the NMS labs, but the benefit from almost eliminating calibration-related uncertainties, through using “precise calibrations”, might be lost without the NMS laboratories.

This section considers what would happen to the accuracy of measurements without the high-quality calibrations that are traceable to the NMS laboratories. That is, this section considers the effect on type-1 errors of swapping “precise calibrations” for “basic calibrations”.

This section sets up a stylised model for the cost of measurement errors when a measurement process is used for conformity testing. The approach taken in this study is to value calibration services in terms of the extent to which high-quality calibration services help firms to reduce mistakes due to measurement errors.

It is important to recognise that conformity testing would still take place, irrespective of the presence of NPL and the calibration labs.

The uncertainty is plus/minus twice the relative standard deviation (RSD), which corresponds to the square root of the variance. The cost of mistakes depends on the relative standard deviation (RSD) of the measurement process (σ). Squaring this relative standard deviation gives the variance of the measurement process (σ^2); and this variance can be split into two parts:

1. The first part comes from sources of error that can’t be removed by calibration. Let ${\sigma_0}^2$ denote the component that isn’t associated with calibration, and this will be the dominant component of the expanded uncertainty.
2. The second part is a component of the expanded uncertainty that can be almost eliminated by high-quality calibration. Let u^2 denote the component that can be removed by using high-quality calibration services.

The components of the expanded uncertainty should be added in quadrature. It follows that the total variance is $\sigma(u)^2 = {\sigma_0}^2 + u^2$, which means the relative standard deviation can be rewritten as:

$$\sigma(u) = \sqrt{{\sigma_0}^2 + u^2} \qquad 26\text{-}1$$

A little further rearrangement then yields:

$$\sigma(u) = \sigma_0 \sqrt{1 + \frac{u^2}{{\sigma_0}^2}}$$

Since u is small relative to σ_0, the relative standard deviation, $\sigma(u)$, can be approximated using a Taylor expansion around $u = 0$:

$$\sigma(u) \approx \sigma_0 + \frac{u^2}{2\sigma_0} \qquad 26\text{-}2$$

We consider some scenarios:

- **Scenario 0**: Businesses use calibrations from the NMS labs or from the top-tier calibration labs that take traceability from the NMS labs. This eliminates any calibration-related uncertainties, meaning that u is zero in this baseline.
- **Scenario 1**: Businesses lose access to high-quality calibrations traceable national standards when the NMS is defunded. Calibration-related uncertainties are no longer eliminated, meaning that: $u > 0$.

Furthermore, as a generally accepted heuristic, any lab engaged in conformance testing aims for a minimum TAR (Test Accuracy Ratio)[42] of 4:1. This implies that the extra uncertainty introduced through less accurate calibrations could be as much as 25% of the baseline uncertainty, which means that $u/\sigma_0 = 25\%$.

Our first assumption is that without a network of calibration labs, which offer services traceable to national standards, many businesses would lose access to high-quality calibrations. Our second assumption is that without access to high quality calibrations, businesses would revert to tests based on this minimum TAR of 4:1. (Whereas, of courses, with access to high quality calibrations, these businesses can eliminate the systematic errors associated with calibration.) Our third assumption is that if the NMS labs ceased to be an anchor for chain of calibrations, then a work-around would be found, which at least preserved a basic kind of calibration. By combining these assumptions, we can estimate the proportional increase in σ that would occur if there wasn't a network of top-tier calibration labs taking traceability from national standards.

Public funding is needed to build up and maintain the capabilities of the NMS labs, which are the used to supply top-quality calibration services. These high-quality calibration services reduce the relative standard deviation of the measurement process, which then reduces the cost of mistakes in conformance testing. (Calibration maybe a small part of the overall measurement spending but it's analogous to the final turn of a screw holding things tight and secure.) Since $u = 0$ in the baseline (Scenario 0), the relative standard deviation is σ_0. Whereas, in Scenario 1 - where business have lost access to high-quality calibrations - the relative standard deviation is $\sigma(u)$. It follows that the change in the relative standard deviation that would occur should businesses lose access to high quality calibrations is given by:

$$\Delta\sigma = \sigma(u) - \sigma_0 \approx \frac{u^2}{2\sigma_0} \qquad 26\text{-}3$$

Hence, the proportional increase in the relative standard deviation becomes:

$$\frac{\Delta\sigma}{\sigma_0} \approx \frac{1}{2}\left(\frac{u}{\sigma_0}\right)^2 \qquad 26\text{-}4$$

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[42] https://www.mitutoyo.com/webfoo/wp-content/uploads/15005A.pdf

Note that the change in σ is relative to the baseline (Scenario 0) in which all calibration-related uncertainties have been eliminated, meaning that: $u = 0$.

Next, it's assumed that without access high-quality calibrations, business default to using a TAR of 4:1, and this implies that $u/\sigma_0 = 25\%$. Substituting this percentage into the previous formula gives:

$$\frac{\Delta\sigma}{\sigma_0} \approx \frac{1}{2}\left(\frac{1}{4}\right)^2 \approx 3.1\% \qquad 26\text{-}5$$

Hence, this analysis implies that without a network of calibration labs taking traceability from national standards, σ would increase by 3.1%. Furthermore, this increase in the relative standard deviation of the measurement process would then lead to more mistakes in conformance testing.

27 THE EFFECT OF TAKING TRACEABILITY FROM A FOREIGN NMI

This section gives a lower bound for the benefit of NMS. In contrast, the previous section gave an upper bound for the benefit of the NMS.

Based on data from the latest NMS survey, only about 14% of domestic customers use a foreign NMI (in conjunction with one or other of the NMS labs). This suggests that, within the UK, the chain of traceability is principally anchored in the NMS labs. Consequently, the central idea in this report is that the organisations currently coming to NPL for calibrations represent that vital first link in the traceability chain; and it's through these organisations that the benefits fanout across the economy. Thus, if some proportion of these organisations ceased to be customers of the NMS labs, then the whole chain of traceability would be weakened. That is, if the first link in the chain is weakened, then the strength of the whole chain is necessarily affected.

This report provides a model of quantifying the benefit created by a national measurement infrastructure, which gives UK-based organisations unimpeded access to precise calibrations anchored in the standards maintained by the NMS labs. Taking traceability from a foreign NMI would affect the chain of calibrations but to some extent the fanout would still exist. Thus, the question remains: How much of this benefit would survive if customers of the NMS labs instead had to depend on services from a foreign NMI?

27.1 WHAT MIGHT HAPPEN WITHOUT THE NMS LABS?

Without the NMS laboratories, the UK's calibration labs would need to go to a foreign National Measurement Institute (NMI) to ensure the traceability of their services. In this situation, it would be more difficult for the labs (and their customers) to access precise calibrations but there are other issues as well:

- Sending instruments abroad would lead to higher costs, as well as longer delays whilst the instruments are in transit and so out of use.
- Unless some kind of deal is made, UK customers might go to the back of the queue in foreign NMIs.
- The inevitable jolting of instruments during the return journey would degrade the precision of the calibration, regardless of how accurate it was when it left the NMI. (For example, consider the jolt received by a sensitive instrument when a plane comes into land and touches down on the runway, or the swaying of cargo in the hold of a boat on a choppy sea.)

Therefore, the frequency and reliability of calibrations would decline somewhat without the primary standards of the NMS laboratories to act as an anchor for the chain of traceability. That is, a form of basic calibration will continue without the NMS, but the benefit that comes from eliminating calibration-related uncertainties would be put in jeopardy without the NMS laboratories.

27.2 LESS USE OF "PRECISE CALIBRATIONS" IF USERS GO TO VSL

By analysing a concrete counterfactual scenario, this subsection arrives at a conservative estimate of the how much the use of precise calibrations would decrease without the NMS labs. Given that VSL in the Netherlands is the closest major NMI to the UK, it would probably be the prime beneficiary of a decision to switch off NPL's services. Thus, we consider a scenario where NPL's funding was cut, and UK-based organisations were encouraged to take traceability from VSL. What proportion of the benefits from access to precise calibrations would be preserved in this scenario?

Currently, NPL sells its services to a wide range of customers from across the world. Indeed, about 50% of its revenue comes from overseas customers, particularly, those based in Europe. This suggests that NPL offers some services that aren't provided by the home NMI of its overseas customers. That is, if their home NMI offered all the services they needed, then they would not need to ship their instruments to the UK so that they can be calibrated at NPL.

Information about sales of NPL's services to both international and domestic customers is collected each year by NPL's finance department. This information has been used to construct an analytical dataset in which the rows (records) are the individual invoices, and the columns give information about the nature of the services, the details of the customers, and when these services were supplied. Specifically, this dataset contains information on the income earned from supplying measurement services, the various countries where NPL's international customers reside, and the year in which each job was delivered.

An econometric analysis (King & Renedo (2020)) was performed using the dataset constructed from NPL's invoicing records. The principal aim of this analysis was to find the relationship between the volume of sales originating from customers in a particular country (number of invoices) and two factors that were thought to influence demand for NPL's services: The price-level ratio between a country and the UK during a given year. This index is a country's purchasing power parity relative to the UK divided by the exchange rate expressed in local currency units per pound.

- The strength of a country's own national measurement institute based on coverage of the Core Measurement Capabilities (CMCs) as detailed in a database maintained by Bureau International des Poids et Mesures (BIPM). This index ranges from 0 to 100, where '100' signifies a full coverage of the CMCs in the BIPM's database and '0' signifies that there's nothing available.

Drawing on the conventional "gravity model" of trade between two countries, our formula also featured a couple of very traditional variables:

- A country's size as represented by its GDP and population.
- The straight line (geodesic) distance between London and a country's largest city. This is the distance in km travelled by a plane flying directly from London to the country's largest city.

This characterisation of international demand for NPL's services was captured by a regression function where the dependent variable was the volume of sales. The regressors were the price-level ratio and the country's characteristics, including the CMC index for the range of services offered by a country's own NMI. Because the dependent variable and most of the regressors entered the formula having been logged, the coefficients in the regression function can generally be interpreted as elasticities.

Based on estimates of the coefficients of key regressors, the headline results were:

- Yearly changes in the UK's price-level ratio with other countries gives an exogenous source of variation in price as experienced by a customer. If the price-level ratio increases by 10%, then demand for NPL's services drops by 12.4%, and so implying an elastic demand for services.

- If a foreign NMI increases its coverage of CMCs on BIPM's database by one percentage point, then demand for NPL's services drops by 2%. Given that the CMC variable ranges from 50% to 75% for the major NMIs, this implies that this CMC variable has a large influence on the demand for NPL's services.
- If the time required to travel between the UK and another country was somehow too double, then demand for NPL's services would drop by 48%. In other words, the coefficient for logged distanced in the regression function is 0.48.

Firstly, in terms of the application of these results to the second scenario under consideration, it is the strong negative relationship between distance and uptake that is the most important. As the estimated coefficient is an elasticity, this negative relationship can be expressed as follows:

$$\frac{\partial I_{it}}{\partial D_i}\frac{D_i}{I_{it}} = -0.48, \qquad \text{27-1}$$

where I_{it} is the number of invoices from customers in country i during year t, and D_i is the distance between London and the largest city in that country. Secondly, although the focus of the econometric analysis was the factors influencing sales to overseas customers, the sample used in the regression analysis also featured invoicing data for sales to UK-based customers. Hence, the effect of switching off NPL's services can be assessed by considering the increase in the distance that UK-based customers would need to ship any instruments that they wanted precisely calibrated.

To assess the impact on existing customers of them having to go overseas for access to precise calibrations, we needed a standard measure of the distance between NPL and its UK-based customers. In trade studies, the conventional formula for a country's internal distance is $D = 0.67\sqrt{A/\pi}$, where A is the geographical area of a country. The geographical area covered by the UK is 244,376 km^2, which implies that its internal distance is 185.8 km.

As discussed at the start of this section, a reasonable counterfactual is that without the NMS labs, customers wanting precise calibrations would need to use VSL in the Netherlands. A simple way to model this shift is to imagine that NPL was bought and relocated to Amsterdam, meaning that NPL's existing UK-based customers would then need to transport their instruments much further to get them precisely calibrated. Whereas the internal distance for UK-based customers is around 186 km, the distance from London to Amsterdam is 357 km. Since it is the logged distance that appears in the regression formula, it's appropriate to estimate the percentage change in the distance as follows:

$$\ln(357\ \text{km}) - \ln(186\ \text{km}) \approx 65\%.$$

Hence, the consequent drop in usage by UK-based customers can be estimated as follows:

$$-0.48 \times [\ln(357\ \text{km}) - \ln(186\ \text{km})] \approx -0.48 \times 65\% \approx -31\%$$

In other words, because the elasticity of invoices with respect to distance is -0.48, increasing the separation of "NPL" and its domestic userbase by 65% would cause demand to drop by 31%. Furthermore, NPL currently covers a higher proportion of the CMCs than VSL, and so prospective customers may encounter some drop in capability if NPL ceased providing its services.[43] Hence, a 31% decrease in the UK's use of precise calibrations is probably a conservative estimate, and so the actual drop could be even larger.

[43] NPL supplies around 75% of the CMCs on BIPM's database against 50% for VSL. This is based on the figures in Table 8 (page 51) of BEIS (2017).

28 WHAT MIGHT HAPPEN IF THE NMS LABS WERE DEFUNDED?

The subsequent analysis will consider a couple of scenarios for what would happen if the NMS labs were defunded. Defunding the NMS labs would probably jeopardise access to "precise calibrations" for most users in the UK. Such users would default to using "basic calibrations" but might also preserve limited access to "precise calibrations" by sending their instruments to be calibrated by a foreign NMI. Let us consider the following scenarios:

- Scenario 1 considers what would happen if users in the UK completely lost access to precise calibrations, because the UKAS labs could only offer "basic calibrations".
- Scenario 2 considers what would happen if users needing "precise calibrations" were signposted to a foreign NMI. In this scenario, the UK retains partial access to "precise calibrations" by taking traceability from a foreign NMI.

In the first scenario, the NMS labs are defunded and there is no alternative source of "precise calibrations" for past users of the NMS labs. (For clarity, note that this is the same "Scenario 1" as in the previous section.) The top-tier calibration labs might send their instruments to foreign NMIs to be recalibrated, but this would not happen very often, and it is just sufficient to backstop a decline in the accuracy of their own calibrations. This ensures that the calibration labs can still offer "basic calibrations" to their customers, but they are no longer able to supply "precise calibrations".

In the second scenario, the NMS labs are defunded but users wanting "precise calibrations" are actively signposted to VSL in the Netherlands. It's been shown that the cost and difficulty of sending instruments abroad for calibration would lead to a drop of 31% in the use of "precise calibration" but access to high-quality calibrations would be maintained.

28.1 SCENARIO 1: USERS LOSE ALL ACCESS TO "PRECISE CALIBRATIONS"

It has already been argued that without access to precise calibrations, UK businesses would revert to using a TAR of 4:1. Moreover, losing access to "precise calibrations" was shown to imply that the relative Standard Deviation (RSD) of the measurements made during conformance tests would increase by about 3.1%. That is, the proportional increase in the RSD is given by:

$$\frac{\Delta\sigma}{\sigma_0} = 3.1\% \qquad \text{28-1}$$

As already discussed, there can be no compromise in the rigour of the test. And, thus, a change in σ leads to a change in the likelihood of type-1 errors; and when the likelihood of type-2 errors is held constant (due to uncompromising demands for rigor), the resulting change in type-1 errors can be written as:

$$\Delta\wp_{1|0} = \left(\frac{\Delta\sigma}{\sigma_0}\right) \times \Phi'\left(\Phi^{-1}\left(1-\wp_{1|0}\right)\right) \times \left\{\Phi^{-1}\left(1-\wp_{1|0}\right) - \Phi^{-1}\left(\wp_{0|1}\right)\right\},$$

where $\wp_{1|0} = 0.3\%$, $\wp_{0|1} = 0.1\%$, and $\Delta\sigma/\sigma_0 = 3.1\%$. Hence, this formula becomes:

$$\Delta\wp_{1|0} = 3.1\% \times \Phi'\left(\Phi^{-1}(99.7\%)\right) \times \left[\Phi^{-1}(99.7\%) - \Phi^{-1}(0.1\%)\right].$$

Evaluating the terms in this expression yields:

$$\Delta\wp_{1|0} = 3.1\% \times 0.00915 \times [2.75 - (-3.09)] = 0.2\% \qquad \text{28-2}$$

Hence, without precise calibrations, $\wp_{1|0}$ goes from 0.3% to 0.5%. In proportional terms, this amounts to an increase of 67% from the baseline. The calculation for this is as follows:

$$\frac{\Delta\wp_{1|0}}{\wp_{1|0}} = \frac{0.2\%}{0.3\%} = 67\% \qquad \text{28-3}$$

Next we consider what happens to $p_{1|0}$ if users can access "precise calibrations" by going to VSL in the Netherlands.

28.2 SCENARIO 2: USERS CAN GET "PRECISE CALIBRATIONS" FROM VSL

It has been shown that if users in the UK had to travel to VSL for "precise calibrations", then the use of "precise calibrations" would drop by at least 31%. The next step is to trace out the consequence of this 31% contraction in uptake for the RSD of measurement made during conformance tests, along with its subsequent effect on the likelihood of type 1 and type 2 errors.

It has already been argued that without access to precise calibrations, UK businesses would revert to using a TAR of 4:1. Moreover, losing access in this way was shown to imply that the relative Standard Deviation (RSD) of the measurements made during conformance tests would increase by about 3.1%. However, this is for the case where the UK lost all access to precise calibrations. Whereas, if the UK could retain access through VSL, then the use of precise calibration would drop by 31%. This reduction in access (as opposed to full outage) would lead the RSD to increase by about one percentage point. The simple calculation is given by:

$$\frac{\Delta\sigma}{\sigma_0} = 31\% \times 3.1\% \approx 1\%. \qquad \text{28-4}$$

As already discussed, there can be no compromise in the rigour of the test. And, thus, a change in σ leads to a change in the likelihood of type-1 errors; and when the likelihood of type-2 errors is held constant (due to an uncompromising demand for rigorous tests), the resulting change in type-1 errors can be written as:

$$\Delta p_{1|0} = \left(\frac{\Delta\sigma}{\sigma_0}\right) \times \Phi'\left(\Phi^{-1}\left(1 - p_{1|0}\right)\right) \times \left\{\Phi^{-1}\left(1 - p_{1|0}\right) - \Phi^{-1}\left(p_{0|1}\right)\right\},$$

where $p_{1|0} = 0.3\%$, $p_{0|1} = 0.1\%$, and $\Delta\sigma/\sigma_0 = 1\%$. Hence, this formula becomes:

$$\Delta p_{1|0} = 1\% \times \Phi'\left(\Phi^{-1}(99.7\%)\right) \times \left[\Phi^{-1}(99.7\%) - \Phi^{-1}(0.1\%)\right].$$

Evaluating the terms in this expression yields:

$$\Delta p_{1|0} = 1\% \times 0.00915 \times [2.75 - (-3.09)] = 0.05\% \qquad \text{28-5}$$

Hence, without precise calibrations, $p_{1|0}$ goes from 0.3% to 0.35%. In proportional terms, this amounts to an increase of 16.7% from the current baseline. The calculation for this is as follows:

$$\frac{\Delta p_{1|0}}{p_{1|0}} = \frac{0.05\%}{0.3\%} = 16.7\% \qquad \text{28-6}$$

So, to summarise: $\Delta p_{1|0} = 0.2\%$ in the first scenario; and $\Delta p_{1|0} = 0.05\%$ in the second scenario. And, having found the change in $p_{1|0}$ under each scenario, this brings Part 4 of our report to an end. The next step is to explore the effect of a change in $p_{1|0}$ on the frequency of inspections.

29 THE EFFECT OF TYPE-1 ERRORS ON THE FREQUENCY OF INSPECTIONS

This section marks the beginning of Part 5 of this report, which explores the effect of changes in the accuracy of measurements on the likelihood of type-1 errors (false positives) in the conformance testing process. Subsequent sections use the results from the two scenarios to conduct a benefit-cost analysis for the NMS programme. However, the next step is to find the effect of a change in $p_{1|0}$ on the optimal frequency of inspections.

Society has the power to set the inspection frequency, n. Moreover, through an iterative process of trial and improvement, society will home in on the frequency that maximises the citizens' prosperity. Thus, the preferred steady state is arrived at by selecting the optimal inspection frequency, $\hat{n}$. If outside factors lead to a change in $p_{1|0}$, then this would naturally alter the optimal inspection frequency.

It has already been established that $\hat{n}$ is optimal if and only if it satisfies equation 22-1. In other words, equation 22-1 is an optimality condition, which yields an implicit equation for $\hat{n}$ once combined with the rest of the equations in the model. It's clear from 22-1 that a change in $p_{1|0}$ must lead to a change in $\hat{n}$. Furthermore, this implicit equation can be used to find the elasticity of $\hat{n}$ with respect to $p_{1|0}$.

Lemma 29-1: *The elasticity of* $\boldsymbol{\hat{n}}$ *with respect to* $\boldsymbol{p_{1|0}}$ *is given by:*

$$\mathcal{E}(\hat{n}, p_{1|0}) = \frac{-\{\mathcal{E}(v^*, \hat{n}) - m^*\}}{\mathcal{R}(v^*, \hat{n}).\mathcal{E}(v^*, \hat{n}) + \{\mathcal{E}(v^*, \hat{n})\}^2 - \{m^*\}^2} \qquad 29\text{-}1$$

Because $\mathcal{E}(v^*, \hat{n}) > m^*$ *and* $\mathcal{R}(v^*, \hat{n}) > 0$*, it follows that* $\mathcal{E}(\hat{n}, p_{1|0}) < 0$*. In other words, the optimal inspection frequency,* $\hat{n}$*, decreases as the likelihood of type-1 errors,* $p_{1|0}$*, increases.*

The proof runs as follows:

Proof. To simplify the notation little, we let $p \equiv p_{1|0}$ so that 22-1 can be written as:

$$\mathcal{E}(v^*, \hat{n}) - m^* - p\hat{n} = 0$$

Differentiating this expression with respect to p gives:

$$\frac{\partial \hat{n}}{\partial p}\frac{\partial}{\partial \hat{n}}[\mathcal{E}(v^*, \hat{n}) - m^*] - \hat{n} - p\frac{\partial \hat{n}}{\partial p} = 0$$

A little rearrangement then gives:

$$\hat{n} = \frac{\partial \hat{n}}{\partial p}\left[\frac{\partial \mathcal{E}(v^*, \hat{n})}{\partial \hat{n}} - \frac{\partial m^*}{\partial \hat{n}} - p\right]$$

From this, the elasticity of $\hat{n}$ with respect to p is given by the following expression:

$$\mathcal{E}(\hat{n}, p) = \frac{p}{\frac{\partial \mathcal{E}(v^*, \hat{n})}{\partial \hat{n}} - \frac{\partial m^*}{\partial \hat{n}} - p}$$

Using the definition of $\mathcal{E}(\cdot)$ we then get:

$$\mathcal{E}(\hat{n}, p) = \frac{p}{\{\mathcal{E}(v^*, \hat{n})/\hat{n}\}\mathcal{E}[\mathcal{E}(v^*, \hat{n})] - \{m^*/\hat{n}\}\mathcal{E}(m^*, \hat{n}) - p}$$

Now, consider the RHS. Multiply the numerator and denominator by $\hat{n}$ to get:

$$\mathcal{E}(\hat{n}, p) = \frac{p\hat{n}}{\mathcal{E}(v^*, \hat{n})\mathcal{E}[\mathcal{E}(v^*, \hat{n})] - m^*\mathcal{E}(m^*, \hat{n}) - p\hat{n}}$$

From equations 18-4 we get: $\mathcal{E}[\mathcal{E}(v^*, \hat{n})] = 1 - \mathcal{E}(v^*, \hat{n}) - \mathcal{R}(v^*, \hat{n})$. And, from equation 19-4, we get: $\mathcal{E}(m^*, \hat{n}) = 1 - m^*$. Substitution of these results into the expression for $\mathcal{E}(\hat{n}, p)$ gives us:

$$\mathcal{E}(\hat{n}, p) = \frac{p\hat{n}}{\mathcal{E}(v^*, \hat{n})[1 - \mathcal{E}(v^*, \hat{n}) - \mathcal{R}(v^*, \hat{n})] - m^*(1 - m^*) - p\hat{n}}$$

Using 22-1 we can rewrite this as follows:

$$\mathcal{E}(\hat{n}, p) = \frac{\mathcal{E}(v^*, \hat{n}) - m^*}{\mathcal{E}(v^*, \hat{n})[1 - \mathcal{E}(v^*, \hat{n}) - \mathcal{R}(v^*, \hat{n})] - m^*(1 - m^*) - [\mathcal{E}(v^*, \hat{n}) - m^*]}$$

After simplifying the denominator this becomes:

$$\mathcal{E}(\hat{n}, p) = \frac{-\{\mathcal{E}(v^*, \hat{n}) - m^*\}}{\mathcal{R}(v^*, \hat{n})\mathcal{E}(v^*, \hat{n}) + \{\mathcal{E}(v^*, \hat{n})\}^2 - \{m^*\}^2}$$

Because $\mathcal{E}(v^*, \hat{n}) > m^*$ and $\mathcal{R}(v^*, \hat{n}) > 0$, it follows that $\mathcal{E}(\hat{n}, p) < 0$. In other words, $\hat{n}$ decreases as p increases. ■

Equation 29-1 can be evaluated using estimates of m^*, $\mathcal{E}(v^*, \hat{n})$, and $\mathcal{R}(v^*, \hat{n})$:[44]

$$\mathcal{E}(\hat{n}, p_{1|0}) = \frac{-\{2.3\% - 2.0\%\}}{207.2\% \times 2.3\% + \{2.3\%\}^2 - \{2.0\%\}^2} = -6.3\%$$

Notice that this is low for an elasticity, which means that $\hat{n}$ responds very inelastically to an increase in $p_{1|0}$. Furthermore, it can be seen, from the basic definition of an elasticity, that multiplying $\mathcal{E}(\hat{n}, p_{1|0})$ by $\Delta p_{1|0}/p_{1|0}$ gives an expression for the percentage change in $\hat{n}$:

$$\frac{\Delta\hat{n}}{\hat{n}} = \mathcal{E}(\hat{n}, p_{1|0}) \times \frac{\Delta p_{1|0}}{p_{1|0}} \qquad 29\text{-}2$$

Moreover, we've just shown that: $\mathcal{E}(\hat{n}, p_{1|0}) = -6.3\%$. However, with respect to $\Delta p_{1|0}/p_{1|0}$, there are two distinct scenarios to consider:

Scenario 1: It's already been shown that if the UK lost all access to precise calibrations (but retained rigorous testing), then $p_{1|0}$ would go from 0.3% to 0.5%, meaning that $\Delta p_{1|0} = 0.2\%$. Which then implies that: $\Delta p_{1|0}/p_{1|0} = 0.2\%/0.3\% = 67\%$. So, this change amounts to a 67% increase in the likelihood of type-1 errors. Based on the expression for $\Delta\hat{n}/\hat{n}$, the corresponding percentage change (decrease) in the frequency of inspections is given by:

$$\frac{\Delta\hat{n}}{\hat{n}} = -6.3\% \times 67\% = -4.2\%$$

Hence, $\hat{n}$ goes from its current value of 1.18 to a lower value of 1.15. This drop in the frequency of inspections equates to going from a gap of 44 weeks between inspections to a gap of 46 weeks between inspections.

Now, consider the situation in which users can access "precise calibrations" by travelling to VSL in the Netherlands. Compared to the first scenario, this leads to a smaller drop in the use of "precise calibrations".

Scenario 2: Using equation 29-1, it has already been shown that the elasticity of $\hat{n}$ with respect to $p_{1|0}$ is $\mathcal{E}(\hat{n}, p_{1|0}) = -6.3\%$. Since $\Delta p_{1|0}/p_{1|0} = 16.7\%$, it follows that:

$$\frac{\Delta\hat{n}}{\hat{n}} = -6.3\% \times 16.7\% = -1.1\%$$

This shows that the frequency of inspections decreases by 1.1%. So, given that $\hat{n} = 1.18$, this implies that: $\Delta\hat{n} = -0.01$. Hence, $\hat{n}$ goes from its current value of 1.18 to a lower value of 1.17. This drop in the frequency of inspections equates to a few extra days between inspections.

A change in the optimal frequency of inspections has implications for the cost of paying CT engineers to supervise production. Recall that τ is the cost of paying CT engineers to inspect one million pounds worth of capital equipment. Furthermore, from equation 10-7, it follows that

[44] It's already been established that $m^* = 2.0\%$, $\mathcal{E}(v^*, \hat{n}) = 2.3\%$, and $\mathcal{R}(v^*, \hat{n}) = 207.2\%$.

$$\tau(\hat{n}) = \frac{\omega}{a}\hat{n}\,,$$

which implies that the cost of supervision, $\tau(\hat{n})$, is proportional to the optimal inspection frequency, $\hat{n}$. Hence, the percentage change in the supervision cost must equal the percentage change in the optimal inspection frequency:

$$\frac{\Delta\tau}{\tau} = \frac{\Delta\hat{n}}{\hat{n}}$$

So, in the first scenario we have $\Delta\tau/\tau = -4.2\%$. And, in the second scenario, we have $\Delta\tau/\tau = -1.1\%$. Given that $\tau = 0.64\%$ in the baseline scenario, it follows that $\tau = 0.61\%$ in the first scenario and $\tau = 0.63\%$ in the second scenario.

Next, consider how a change in the frequency of inspections will affect the equilibrium rental rate. Since the equilibrium rental rate, r^*, depends partly on the cost of supervision, τ, it follows that a change in the optimal frequency of inspections will feed through to a change in the equilibrium rental rate. Furthermore, from equation 17-13, it follows that:

$$r^*(\hat{n}) = \frac{\beta}{s}[\delta + g + s\tau(\hat{n})].$$

Which then implies that the change in the equilibrium rental rate is given by:

$$\Delta r^* = \beta\Delta\tau = (\beta\tau)(\Delta\tau/\tau),$$

where $\beta\tau = 31\% \times 0.64\% =$ corresponds to the part of the supervision cost that is paid by capital. It can be shown that $\beta\tau = 31\% \times 0.64\% = 0.20\%$, from which we get $\Delta r^* = -0.008\%$ in the first scenario and $\Delta r^* = -0.002\%$ in the second scenario. Given that $r^* = 9.96\%$, these shifts in the equilibrium rental rate are negligible, and so can be ignored for practical economic purposes. In other words, theoretically, a drop in the optimal frequency of inspections does lead to a drop in the equilibrium rental rate, but this effect is so small that it's not worth further consideration. So, for practical economic purposes, we can continue to regard r^* as an immutable constant of the system.

The final step is to consider how a change in the frequency of inspections will affect the proportion of GVA that is spent on conformance testing. Given that $m^* = 2\%$ and the GVA of the real economy is £1.2 trillion, the spend on conformance testing must be £24 billion. However, a decrease in $\hat{n}$, due to an increase in $\wp_{1|0}$, would necessarily lead to a decrease in m^*. That is, a drop in the frequency of inspections naturally leads to a decrease in the amount spent on conformance testing.

Let Δm^* denote the change in m^* that would occur if the UK lost all access to precise calibrations. The basic definition of an elasticity then implies the percentage change in m^* is given by:

$$\frac{\Delta m^*}{m^*} = \mathcal{E}(m^*, \hat{n}) \times \frac{\Delta\hat{n}}{\hat{n}} \qquad \text{29-3}$$

From 19-4, we have $\mathcal{E}(m^*, \hat{n}) = 1 - m^*$. And, since $m^* = 2\%$, this then implies that $\mathcal{E}(m^*, \hat{n}) = 100\% - 2\% = 98\%$. With respect to the change in $\hat{n}$, there are two scenarios to consider:

Scenario 1: It's been shown that with the loss of precise calibrations would cause the frequency of inspections to drop by 4.2%. Hence, the percentage change in the proportion of GVA spent on conformance testing is given by:

$$\frac{\Delta m^*}{m^*} = 98\% \times (-4.2\%) = -4.1\%$$

If the UK spends £28.8 billion each year on conformance testing, then a drop of 4.1% in this testing activity implies that such spending will decrease by £1.2 billion. However, the decrease in the cost to businesses is also given by: $\omega\Delta E^{*} = -£1.2\ \text{billion},$ where ΔE^{*} is the decrease in the number of CT engineers employed in the economy, and ω is their wage rate. Since $\omega = £32.4$ thousand, this result implies that: $\Delta E^{*} = -37$ thousand. The expectation is that these engineers will be reemployed in other roles but will no longer be doing conformance testing.

Now, consider the change in CT spending in the situation where users are signposted to VSL in the Netherlands if they need "precise calibrations". In this situation, the drop in CT spending won't be as large as in the first scenario where users lose all access to "precise calibrations".

Scenario 2: Under the second scenario, the frequency of inspections decreases by 1.1%, which leads to a similar dop in the amount spent on conformance testing. So, given that the current spending on conformance testing amounts to £28.8 billion each year, and a decrease in testing activity of 1.1% yields around £317 million in savings. However, the decrease in the cost to businesses is also given by: $\omega\Delta E^{*} = -£317\ \text{million},$ where ΔE^{*} is the decrease in the number of CT engineers employed in the economy, and ω is their wage rate. Furthermore, since $\omega = £32.4$ thousand, this result implies that: $\Delta E^{*} = -9.8$ thousand. Lastly, note that these engineers will be reemployed in other engineering roles but will no longer be doing conformance testing.

The next step is to consider how changes in $\wp_{1|0}$ and $\hat{n}$ will affect the reliability of production processes.

30 THE EFFECT OF TYPE-1 ERRORS ON THE RELIABILITY OF PRODUCTION

So far, this analysis has established that an increase in $\wp_{1|0}$ will lead to a decrease in $\hat{n}$. The next step is to trace out the consequence of these changes on other parameters and variables.

θ and ϕ depend on the frequency of inspections and the likelihood of type-1 and type-2 errors. Hence, changes in $\hat{n}$ and $p \equiv \wp_{1|0}$ feed through to changes in θ and ϕ. Let Δθ and Δϕ denote the resulting changes in θ and ϕ, respectively.

Lemma 30-1: It can be shown that the total derivatives of **θ** and **ϕ** are given by:

$$\Delta\theta = (1-\theta)(\hat{n}\Delta p + p\Delta\hat{n})\,, \qquad (30\text{-}1)$$

$$\Delta\phi = (1-\phi)(\hat{n}\Delta q + q\Delta\hat{n})\,, \qquad (30\text{-}2)$$

where $p \equiv \wp_{1|0}$ and $q \equiv \wp_{1|1}$.

Proof. Since θ is a function of $p \equiv \wp_{1|0}$ and $\hat{n}$, the total derivative of θ is given by:

$$\Delta\theta = \frac{\partial\theta}{\partial p}\Delta p + \frac{\partial\theta}{\partial\hat{n}}\Delta\hat{n}\,,$$

where $\partial\theta/\partial p = (1-\theta)\hat{n}$ and $\partial\theta/\partial\hat{n} = (1-\theta)p$. Hence, we arrive at:

$$\Delta\theta = (1-\theta)(\hat{n}\Delta p + p\Delta\hat{n}).$$

A very similar analysis gives the total derivative of Δϕ. ■

The expressions for $\Delta\theta$ and $\Delta\phi$ can be evaluated using estimates of the quantities involved, and we've already seen that there are two scenarios to consider. Recall that in both scenarios the CT testing regime remains rigorous, which means there can't be any change in $q \equiv \wp_{1|1}$. In other words, we must have $\Delta q = 0$ in both scenarios.

Scenario 1: If users lost all access to precise calibrations, then the estimates of $\Delta\theta$ and $\Delta\phi$ are as follows:

$$\Delta\theta = (1 - 0.3\%) \times [1.18 \times 0.2\% + 0.3\% \times (-0.03)] \approx 0.2\%$$

$$\Delta\phi = (1 - 69.4\%) \times [1.18 \times 0\% + 99.9\% \times (-0.03)] \approx -0.9\%$$

So, if the UK lost access to precise calibrations, then θ goes from 0.3% to 0.5% and ϕ goes from 69.4% to 68.5%. Thus, the "regret rate" increases and the "detection rate" decreases, implying a deterioration in the effectiveness testing process.

Now, consider the second scenario in which users can access "precise calibrations" from VSL in the Netherlands.

Scenario 2: If users retained access to "precise calibrations" from VSL in the Netherlands, then the estimates of $\Delta\theta$ and $\Delta\phi$ are as follows:

$$\Delta\theta = (1 - 0.3\%) \times [1.18 \times 0.05\% + 0.3\% \times (-0.01)] \approx 0.06\%$$

$$\Delta\phi = (1 - 69.4\%) \times [1.18 \times 0\% + 99.9\% \times (-0.01)] \approx -0.3\%$$

So, if users needed to go to VSL for "precise calibrations", then θ goes from 0.30% to 0.36% and ϕ goes from 69.4% to 69.1%. Thus, the "regret rate" increases and the "detection rate" decreases, implying a deterioration in the effectiveness of the testing process.

Since v^* depends on the "detection rate", ϕ, it follows that a change in ϕ will lead to a change in the "detection rate". Let Δv^* denote the change in v^* that comes from losing access to precise calibrations.

Lemma 30-2: It can be shown that the change in the reliability of production, $\boldsymbol{v}^*$, that comes from a change in the "detection rate", $\boldsymbol{\phi}$, is given by the following expression:

$$\Delta v^* = \frac{v^*(1 - v^*)\Delta\phi}{\delta + g + \phi} \tag{30-3}$$

The proof runs as follows:

Proof. It has already been shown that the equilibrium condition for $dv/dt = 0$ is as follows:

$$(1 - v^*)(\delta + g + \phi) = \varepsilon v^*$$

(The LHS represents the outflow from $1 - v^*$ and the RHS represents the inflow to $1 - v^*$. In equilibrium, the inflow and the outflow must equal one another.) Differentiating this expression with respect to ϕ gives:

$$-\frac{\partial v^*}{\partial\phi}(\delta + g + \phi) + (1 - v^*) = \varepsilon\frac{\partial v^*}{\partial\phi}$$

Solving for $\partial v^*/\partial\phi$ yields:

$$\frac{\partial v^*}{\partial\phi} = \frac{(1 - v^*)}{\delta + g + \phi + \varepsilon}$$

Since $v^* = (\delta + g + \phi)/(\delta + g + \phi + \varepsilon)$, this can be rewritten as:

$$\frac{\partial v^*}{\partial\phi} = \frac{v^*(1 - v^*)}{\delta + g + \phi}$$

Lastly, using $\Delta v^* := (\partial v^*/\partial\phi)\Delta\phi$, we arrive at the main result. ■

The expression for Δv^* can be evaluated by substituting in estimates of the quantities involved. There are two scenarios to consider:

Scenario 1: Evaluating the expression for Δv^* using estimates from Scenario 1, yields:

$$\Delta v^* = \frac{95.1\% \times (1 - 95.1\%) \times (-0.9\%)}{6.3\% + 69.4\%} \approx -0.1\%$$

So, if the UK lost access to "precise calibrations", then v^* would go from 95.1% to 95.0%. This implies a negligible drop in the reliability of production processes.

Now, consider the situation where users can access precise calibration by going to VSL in the Netherlands. The drop in v^* will be even smaller in this situation than under Scenario 1.

Scenario 2: Evaluating the expression for Δv^* using estimates from Scenario 2, yields:

$$\Delta v^* = \frac{95.1\% \times (1 - 95.1\%) \times (-0.3\%)}{6.3\% + 69.4\%} \approx -0.02\%$$

So, if users had to go to VSL for "precise calibrations", then v^* would go from 95.1% to 95.08%. This implies a very negligible drop in the reliability of production processes.

The changes in v^*, ϕ, and θ feed through to changes in the scrap rate and the rebate rate. The various estimates found above can now be used to evaluate the scrap rate and the rebate rate.

Scenario 1: Under scenario 1, the calculations are as follows:

$$\text{scrap rate} = \theta v^* + \phi(1 - v^*) = 0.5\% \times 95\% + 68.5\% \times (1 - 95\%) \approx 3.9\%$$

$$\text{rebate rate} = (1 - \phi)(1 - v^*) = (1 - 68.5\%)(1 - 95\%) \approx 1.6\%$$

This shows that the scrap rate increases from 3.7% to 3.9%; and the rebate rate increases from 1.5% to 1.6%. This implies that if the UK were to lose access to precise calibrations, then this would lead to a noticeable increase in the scrap rate, along with a modest increase in the rebate rate. In short, the consequences of losing access to precise calibrations would become most evident in a higher scrap rate.

Now, consider the second scenario in which users can go to VSL for "precise calibrations".

Scenario 2: Under scenario 2, the calculations are as follows:

$$\text{scrap rate} = \theta v^* + \phi(1 - v^*) = 0.36\% \times 95\% + 69.1\% \times (1 - 95\%) \approx 3.8\%$$

$$\text{rebate rate} = (1 - \phi)(1 - v^*) = (1 - 69.1\%)(1 - 95\%) \approx 1.55\%$$

This shows that the scrap rate increases from 3.7% to 3.8%; and the rebate rate increases from 1.50% to 1.55%. This implies that if users had to go to VSL for "precise calibrations", then this would lead to a noticeable increase in the scrap rate, along with a very small increase in the rebate rate.

31 THE EFFECT OF TYPE-1 ERRORS ON CAPITAL INTENSITY

Recall that net revenue per capita, $y_{\dagger}^*$, is proportional to the economy's capital intensity in the steady state, k^*. So, to find the long-run effect of a shift in the likelihood of type-1 errors, $\wp_{1|0}$, on the economy, we need to understand the relationship between $\wp_{1|0}$ and k^*.

As already discussed, the system is composed of reliability v and capital intensity k. This two-variable system will settle into a steady-state, where the equilibrium (v^*, k^*) depends on the model's parameters. In particular, the equilibrium depends on the likelihood of type-1 errors, $\wp_{1|0}$. That is, a different choice of parameter values will lead to different equilibrium values for reliability and capital intensity.

Society has the power to set the inspection frequency, n, and through an iterative process of trial and improvement society will home in on the frequency that maximises the citizens' prosperity. Thus, the preferred steady state is that which maximises equilibrium capital intensity, k^*, and is arrived at by selecting the optimal inspection frequency, $\hat{n}$.

If outside factors lead to a change in $\wp_{1|0}$, then this would alter the optimal inspection frequency, which would then feed through to a change in the capital intensity. Hence, this section uses an "envelope argument" to analyse the effect of small changes in $\wp_{1|0}$ on the economy's capital intensity in the resulting equilibrium.

The Envelope Theorem: Let this technique from comparative statics be explained using a generic function: Suppose $\Psi(x|b)$ is a generic function of x, and b is some parameter. Let $\hat{x}(b)$ denote the optimal value of x, so that $\Psi(\hat{x}(b)|b)$ is the maximum attainable value of Ψ. By definition, $\hat{x}(b)$ satisfies the first-order condition for the optimum: $\Psi_x(\hat{x}(b)|b) = 0$, where $\Psi_x(x|b) \equiv \partial\Psi/\partial x$. Next, differentiation of $\Psi(\hat{x}(b)|b)$ with respect to b gives:

$$\frac{d\Psi(\hat{x}(b)|b)}{db} = \frac{d\hat{x}(b)}{db}\frac{\partial\Psi(x|b)}{\partial x}\bigg|_{x=\hat{x}(b)} + \frac{\partial\Psi(x|b)}{\partial b}\bigg|_{x=\hat{x}(b)}$$

But, from the first-order condition for the maximum, we already know that:

$$\frac{\partial\Psi(x|b)}{\partial x}\bigg|_{x=\hat{x}(b)} = \Psi_x(\hat{x}(b)|b) = 0$$

Hence, the change in $\Psi(\hat{x}(b)|b)$ resulting from a change in b can be found by evaluating the partial derivative of $\Psi(x|b)$ at the optimal value of x:

$$\frac{d\Psi(\hat{x}(b)|b)}{db} = \frac{\partial\Psi(x|b)}{\partial b}\bigg|_{x=\hat{x}(b)}$$

The envelope argument can be applied to the economy's equilibrium capital intensity, k^*, by noting the following points:

- $p \equiv \wp_{1|0}$ is a parameter that's exogenously determined by factors outside the model.
- n is under society's control, and so will be chosen to maximise the economy's equilibrium capital intensity, k^*. Hence, the inspection frequency will be set at the socially optimal level, $\hat{n}$, in response to a given value of p, making it a function of p.
- $\hat{n}(p)$ signifies that the optimal inspection frequency is a function of p. We can think of $\hat{n}(p)$ as being set by a social planner who aims to maximise the economy's equilibrium capital intensity.

Let $Q(n|p)$ be the economy's capital intensity in equilibrium, expressed as a function of n and p. So, the maximum attainable capital intensity is $Q(\hat{n}(p)|p)$, where $\hat{n}(p)$ denotes the optimal inspection frequency. Therefore, from the envelope argument, we have the following useful relation:

$$dQ(\hat{n}(p)|p)/dp = \{\partial Q(n|p)/\partial p\}|_{n=\hat{n}(p)} \qquad \text{31-1}$$

This means that finding the change in capital intensity involves partially differentiating $\boldsymbol{Q(n|p)}$ with respect to $\boldsymbol{p}$ and then evaluating the result at $\boldsymbol{n = \hat{n}(p)}$. Using the basic definition of the elasticity function $\mathcal{E}(\cdot)$, the RHS of equation 31-1 can be written in terms of the elasticity of $\boldsymbol{k}^*$ with respect to $\boldsymbol{p}$:

$$\frac{d}{dp}\{Q(\hat{n}(p)|p)\} = \mathcal{E}[Q(\hat{n}(p)|p), p] \times Q(\hat{n}(p)|p)/p \qquad 31\text{-}2$$

Next, the economy is assumed to settle into the "preferred equilibrium", in which equilibrium capital intensity attains its maximum possible value:

$$k^*(\hat{n}(p)) = Q(\hat{n}(p)|p)$$

Furthermore, the analysis in section 0 found that, at the "preferred equilibrium", the values of p, $\hat{n}(p)$, and $k^*(\hat{n}(p))$ are as follows: $p = 0.3\%$; $\hat{n}(p) = 1.18$; $k^*(\hat{n}(p)) =$ £141.1 thousand.

Let Δp denote a small change in p, where p represents the current likelihood of making a type-1 error, and $p + \Delta p$ represents the new likelihood of making a type-1 error. The change in k^* due to this change in p is given by:

$$\Delta k^* = Q(\hat{n}(p + \Delta p)|p + \Delta p) - Q(\hat{n}(p)|p)$$

Furthermore, based on the basic concept of a derivative, this is equivalent to:

$$\Delta k^* = \frac{d}{dp}\{Q(\hat{n}(p)|p)\} \times \Delta p$$

As already discussed, the derivative $\frac{d}{dp}\{Q(\hat{n}(p)|p)\}$ can be found using an envelope argument. In other words, using equation 31-2, the previous expression for Δk^* can be rewritten as:

$$\Delta k^* = \mathcal{E}[Q(n|p), p]|_{n=\hat{n}(p)} \times Q(\hat{n}(p)|p) \times \Delta p/p$$

Since the "preferred equilibrium" is such that $k^*(\hat{n}(p)) = Q(\hat{n}(p)|p)$, this relationship can be expressed neatly in terms of a ratio of the corresponding percentage changes:

$$\frac{\Delta k^*}{k^*} = \mathcal{E}[Q(n|p), p]|_{n=\hat{n}(p)} \times \frac{\Delta p}{p} \qquad 31\text{-}3$$

Hence, the next step is to find an expression for this elasticity:

Lemma 31-1: *$\boldsymbol{n}$ is the frequency of inspections under the conformance testing regime; and $\boldsymbol{p} \equiv \boldsymbol{p}_{\mathbf{1|0}}$ denotes the likelihood of making a type-1 error. Let $\boldsymbol{Q}(\boldsymbol{n}|\boldsymbol{p})$ be the economy's capital intensity, in equilibrium, expressed as a function of $\boldsymbol{n}$ and $\boldsymbol{p}$. The elasticity* of $\boldsymbol{Q}(\boldsymbol{n}|\boldsymbol{p})$ *with respect to $\boldsymbol{p}$ is given by $\mathcal{E}[\boldsymbol{Q}(\boldsymbol{n}|\boldsymbol{p}), \boldsymbol{p}]$, where $\mathcal{E}(\cdot)$ was defined by equation* 18-1*. It can be shown that:*

$$\mathcal{E}[Q(n|p), p] = -\frac{p.n}{(1-\beta)} \qquad 31\text{-}4$$

where β is the share of economic output that goes to the owners of capital.

To connect this Lemma to equation 31-3, it remains to evaluate the elasticity encapsulated by equation 31-4 at the optimal inspection frequency: $\hat{n}(p)$ is society's optimal inspection frequency, expressed as a function of the likelihood of making a type-1 error, p. The optimal inspection frequency maximises equilibrium capital intensity, meaning that:

$$k^*(\hat{n}(p)) = Q(\hat{n}(p)|p) \qquad 31\text{-}5$$

So, evaluating both sides of equation 31-4 at this social optimum gives:

$$\mathcal{E}[Q(n|p), p]|_{n=\hat{n}(p)} = -\frac{p.\hat{n}(p)}{(1-\beta)} \qquad 31\text{-}6$$

Where the proof of Lemma 31-1 runs as follows:

Proof. The likelihood of type-1 errors is $p \equiv \wp_{1|0}$, and this parameter enters our model through the expression for the regret rate: $\theta = 1 - \exp(-p n)$, where n is the frequency of inspections under the conformance testing regime. However, p does not feature in the expressions for ϕ and τ. It follows that:

$$\frac{\partial\theta}{\partial p} = (1-\theta)n;\ \frac{\partial\phi}{\partial p} = 0;\ \frac{\partial\tau}{\partial p} = 0.$$

Hence, the false positives rate increase as the likelihood of type-1 errors increases, whilst the detection rate and the supervision cost are unchanged. So, the key thing to note is that p occurs in the formula for θ, whereas p does not feature in the formulae for ϕ or τ.

Next, as has already been shown, the steady state of the system is characterised by the following formulae for the equilibrium values of v and k:

$$v^* = \frac{\delta + g + \phi}{\delta + g + \phi + \varepsilon}$$

$$s(1-\theta)v^* f(k^*) = (\delta + g + s\tau)k^*$$

Notice that v^* is independent of p because θ doesn't feature in the first formula, which implies $\partial v^*/\partial p = 0$. Hence, all the action is in the second formula and v^* can be treated as a constant. Taking logs of the second formula and differentiating with respect to p gives:

$$-\frac{1}{(1-\theta)}\frac{\partial\theta}{\partial p} + \frac{f_k(k^*)}{f(k^*)}\frac{\partial k^*}{\partial p} = \frac{1}{k^*}\frac{\partial k^*}{\partial p}$$

Since $f_k(k^*)k^* = \beta f(k^*)$, this then becomes:

$$-\frac{1}{(1-\theta)}\frac{\partial\theta}{\partial p} + \frac{\beta}{k^*}\frac{\partial k^*}{\partial p} = \frac{1}{k^*}\frac{\partial k^*}{\partial p}$$

A little further rearrangement then gives:

$$-\frac{1}{(1-\theta)}\frac{\partial\theta}{\partial p} = (1-\beta)\frac{1}{k^*}\frac{\partial k^*}{\partial p}$$

Finally, substituting for $\partial\theta/\partial p$ leads to an expression for the elasticity of k^* with respect to p:

$$\frac{p}{k^*}\frac{\partial k^*}{\partial p} = -\frac{p.n}{(1-\beta)}$$

Which using the definition of $\mathcal{E}(\cdot)$ can then be expressed as:

$$\mathcal{E}(k^*, p) = -\frac{p.n}{(1-\beta)}$$

This completes the proof. ■

Combining equations 31-3 and 31-6 gives:

$$\frac{\Delta k^*}{k^*} = -\frac{\Delta p.\hat{n}(p)}{(1-\beta)} \qquad \text{31-7}$$

This equation can be used to find the percentage decrease in capital intensity that is caused by an increase in the likelihood of type-1 errors (false positives). It has already been established that $\hat{n}(p) = 1.18$ and $1 - \beta = 69\%$. However, with respect to Δp, there are two scenarios to consider:

Scenario 1: Section 25 found that if users lost all access to "precise calibrations", then the likelihood of type-1 errors would increase from 0.3% to 0.5%. Hence, under Scenario 1, we get: $\Delta p = 0.2\%$. Substituting this value for Δp into equation 31-7, yields:

$$\frac{\Delta k^*}{k^*} = -\frac{0.2\% \times 1.18}{69\%} = -0.34\%$$

Hence, without access to "precise calibrations", the economy's capital intensity would contract by 0.34%.

Now, consider the situation in which users can access precise calibration by going to VSL in the Netherlands. The drop in capital intensity won't be as large as under Scenario 1.

Scenario 2: Using Lemma 31-1, it has already been shown that an increase in the likelihood of type-1 errors leads to a reduction in capital intensity. Under Scenario 2, the likelihood of type-1 errors increases from 0.3% to 0.35%, and so we get: $\Delta p = 0.05\%$. Substituting our estimate of $\Delta \wp_{1|0}$ into equation 31-7 implies that:

$$\frac{\Delta k^*}{k^*} = -\frac{\Delta p \times \hat{n}}{(1-\beta)} = -\frac{0.05\% \times 1.18}{69\%} \approx -0.09\%$$

Hence, if users had to access to "precise calibrations" through VSL, then capital intensity would drop by around 0.09%.

This analysis has enabled us to quantify the effect on equilibrium capital intensity of using CT tests that are more prone to false positives because of a cut in the NMS. However, the underlying economics are somewhat obscured by the calculations required. So, before moving on to consider the effects on GVA, it's helpful to summarise the economic argument for why there's a decline for in capital intensity.

31.1 THE ECONOMICS OF THIS FALL IN CAPITAL INTENSITY

This subsection brings together various elements from the preceding analysis and uses it to give a step-by-step explanation of why there would be a fall in capital intensity if the NMS labs were shutdown.

Many businesses directly (or indirectly) depend on the "precise calibrations" that are traceable to the highly accurate standards maintained by the NMS labs. Without the NMS labs, access to these "precise calibrations" would be stopped, or much reduced, so that many businesses default to using "basic calibrations". This would reinsert calibration-related uncertainties back into the measurements made during testing. Consequently, the relative standard deviation (RSD) of such measurements would increase relative to the existing baseline, where calibration-related uncertainties are almost eliminated. There are two scenarios to consider:

- **Scenario 1**: If the NMS labs were shut down without arranging for any alternative source of "precise calibrations", then the RSD of the conformance tests would increase by 4%. This decline in accuracy is the result of UK-based businesses defaulting to the use of "basic calibrations".
- **Scenario 2**: If the NMS labs were shut down after first arranging for those needing "precise calibrations" to use VSL in the Netherlands, then the RSD of the businesses' conformance tests would increase by 1%. This decline in accuracy originates from the extra hassle of having to sending instruments abroad, which reduces the demand for "precise calibrations" amongst UK-based businesses.

The likelihood of type-1 errors in the tests (false positives) is denoted by $p \equiv \wp_{1|0}$. A change in the RSD of the test feeds through to a change in the likelihood of type-1 errors. Let Δp denote the change in p that comes from the change in the RSD. The reason for a rise in false positives is that the tests should remain rigorous, despite an increase in the RSD of the test, which means that the likelihood of a type-2 error is being held constant to prevent a loss of confidence amongst buyers and sellers. Consequently, the extra uncertainty, that was introduced by a rise in the RSD, can only be absorbed by an increase in the likelihood of type-1 errors.

Let Δk^* denote the change in equilibrium capital intensity resulting from a change in the likelihood of type-1 error. From the basic concept of a derivative, this change in k^* can be written as follows:

$$\Delta k^* = \frac{dk^*}{dp}\Delta p$$

where Δp is the change in the likelihood of type-1 errors. This can be rewritten in terms of the percentage changes:

$$\frac{\Delta k^*}{k^*} = \left(\frac{p}{k^*}\frac{dk^*}{dp}\right) \times \frac{\Delta p}{p}$$

The optimal inspection frequency, $\hat{n}$, depends on the likelihood of type-1 errors, p. Hence, the chain rule of differentiation gives us:

$$\frac{dk^*}{dp} = \left.\frac{\partial k^*}{\partial p}\right|_{n=\hat{n}} + \left(\frac{d\hat{n}}{dp} \times \left.\frac{\partial k^*}{\partial \hat{n}}\right|_{n=\hat{n}}\right)$$

Notice that these derivatives are evaluated at the optimal inspection frequency: $n = \hat{n}$. Moreover, because the optimal inspection frequency, $\hat{n}$, maximises the equilibrium capital intensity, k^*, it follows that:

$$\left.\frac{\partial k^*}{\partial \hat{n}}\right|_{n=\hat{n}} = 0$$

Through this first order condition, the Envelope Theorem gives us:

$$\frac{dk^*}{dp} = \left.\frac{\partial k^*}{\partial p}\right|_{n=\hat{n}}$$

Which then implies:

$$\Delta k^* = \Delta p \times \left.\frac{\partial k^*}{\partial p}\right|_{n=\hat{n}}$$

So, the percentage change in equilibrium capital intensity becomes:

$$\frac{\Delta k^*}{k^*} = \left(\frac{p}{k^*}\left.\frac{\partial k^*}{\partial p}\right|_{n=\hat{n}}\right) \times \frac{\Delta p}{p}$$

Next, recall that the equilibrium capital intensity is given by:

$$k^* = \left(\frac{\beta B^*}{r^*}\right)^{1/\alpha},$$

where the effective TFP is $B^* = (1-\theta)v^*A$, and the rental rate is $r^* = (\beta/s)(\delta + g + s\tau)$. Recall that p enters the model through the "regret rate", θ, which means that the effective TFP depends on p, whereas r^* is independent of p. So, from the chain rule of differentiation, it follows that:

$$\frac{\partial k^*}{\partial p} = \frac{\partial k^*}{\partial B^*}\frac{\partial B^*}{\partial p}$$

Furthermore, because the reliability of production processes, v^*, does not depend on p or θ, it follows that:

$$\frac{\partial k^*}{\partial p} = \frac{\partial k^*}{\partial B^*}\frac{\partial B^*}{\partial \theta}\frac{\partial \theta}{\partial p}$$

From the well-established formulae for k^*, B^*, and θ, we have the following derivatives:

$$\frac{\partial k^*}{\partial B^*} = \frac{1}{\alpha}\frac{k^*}{B^*};\ \frac{\partial B^*}{\partial \theta} = -\frac{B^*}{(1-\theta)};\ \frac{\partial \theta}{\partial p} = n(1-\theta).$$

Substituting these formulae into the previous expression for $\partial k^*/\partial p$, whilst also setting n to $\hat{n}$, gives:

$$\left.\frac{\partial k^*}{\partial p}\right|_{n=\hat{n}} = -\frac{\hat{n}k^*}{\alpha}$$

Finally, substituting this result into our earlier expression for the percentage change in equilibrium capital intensity, yields:

$$\frac{\Delta k^*}{k^*} = -\frac{\hat{n}\Delta p}{\alpha}$$

This is really the main headline result of this section. However, before moving on to consider the effects on GVA, it’s helpful to give a graphical explanation for why there’s a decline for in capital intensity.

31.2 A GRAPHICAL EXPLANATION OF THE FALL IN CAPITAL INTENSITY

This subsection gives graphical account of why there would be a decline for in capital intensity following a cut in the NMS. The idea is that less reliable measurements mean that conformance testing becomes less effective, because of a rise in “false positives”. This decline in the effectiveness of the tests leads to a higher scrap rate, which then reduces the efficiency of the production process.

The figure below gives a graphical representation of the effect of a change in the effectiveness of conformance testing on the equilibrium capital intensity. The horizontal axis is capital intensity, and the vertical axis is the marginal product of capital (MPK).

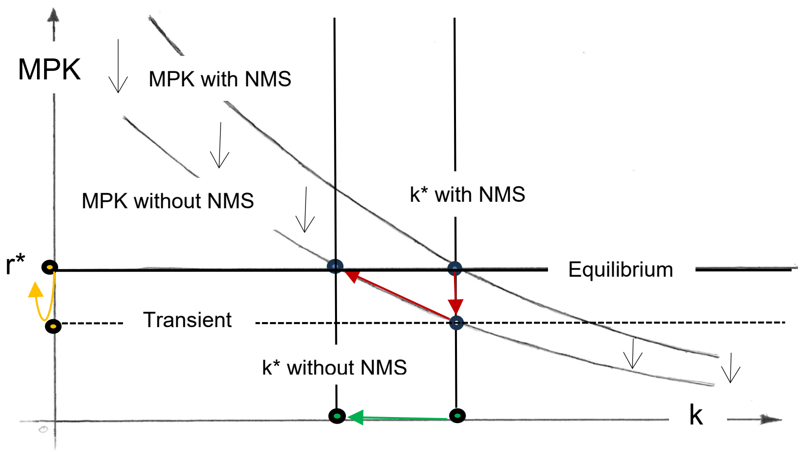

Figure 7: A dop in the marginal product of capital (MPK)

The curved lines represent the MPK at varying levels of capital intensity. These curves are downward sloping because, as capital becomes more plentiful, its marginal product declines. The top curve represents the MPK with effective conformance testing underpinned by accurate calibrations traceable to the NMS labs. The lower curve represents the downward shift in the MPK that occurs without the NMS labs (and even if users can go to a foreign NMI).

A fall in the effectiveness of conformance testing, along with a consequent decrease in the frequency of testing, means that the production becomes less efficient. Specifically, conformance testing suffers from an increase in the rate of false positives, resulting in a higher scrap rate, and so a decline in the economy's effective TFP. Secondly, along with this fall in TFP there also comes a fall in the return on capital, as machines become less productive, meaning that there is a drop in the marginal product of capital. In other words, the curve representing the demand for capital shifts downwards, as illustrated by the figure above.

Capital is best thought of as a "state" variable, because it's value can't change instantaneously and is fixed by what happened in the past. (This contrasts with a "control", such as, consumption, which can change discontinuously from one year to the next.) Moreover, because capital is a "state" variable, its short-run supply is highly inelastic, to the point of almost being a fixed quantity. Consequently, the short-run supply is best represented by a vertical line that can move left or right, depending on whether net-investment is negative or positive. In the diagram, there are two important vertical lines, each representing the capital intensity of the economy in two different equilibriums: The righthand line is the supply of capital in the existing baseline equilibrium, whereas the lefthand line is the supply of capital in the new equilibrium, established following a fall in the marginal product of capital (MPK).

There are two horizontal lines representing the rental rate for capital in the long-run and the short-run. The top line corresponds to the rental rate when the economy is in the baseline equilibrium, implying that the rental rate is: $r^* = (\beta/s)(\delta + g + s\tau)$. Recall that r^* is close to being a fixed constant of the system, and the rental rate, $r(t)$, will always return to this value even if the system is knocked out of equilibrium by a shock. The (doted) bottom line corresponds to the rental rate immediately after a sudden fall in the marginal product of capital (MPK). Because the supply of capital is fixed in the short run, there's a temporary crash in the price of capital as system adjust to a lower MPK. This sudden fall in the rental rate is represented by the red downward arrow. However, subsequent the drop in investment means that capital intensity naturally begins to fall (due to growth in the workforce and the depreciation of capital items). Consequently, over a period of a few years, the rental rate for capital will follow the upward diagonal arrow towards the new steady state. That is, in the long run, $r(t)$ will always return to r^*.

This analysis illustrates how the system re-establishes an equilibrium following a fall in the MPK. However, it's important to recognise that the new equilibrium clearly represents a somewhat smaller, poorer economy than the one before the fall precipitated by a cut in the NMS.

31.3 THE EFFECT OF A LOWER CAPITAL INTENSITY ON ECONOMIC OUTPUT

The next step is to consider the effect of the change in capital intensity on citizens' prosperity.

It has already been shown that, in equilibrium, savings have to equal investment, yielding an equation for the circular follow of money. (The "circular flow" equation came from combining the equilibrium condition for capital intensity with the equation for capital's share of output.) Hence, the net revenue per capita, $y_\dagger^*$, is proportional to capital intensity, k^*:

$$y_\dagger^* \propto k^*$$

This proportionality then implies that the percentage change in one variable must equal the percentage change in the other variable:

$$\frac{\Delta y_\dagger^*}{y_\dagger^*} = \frac{\Delta k^*}{k^*} \qquad 31\text{-}8$$

where $\Delta y_\dagger^*$ is the change in $y_\dagger^*$ which occurs because of the change in k^*. Note that the size of the UK's workforce is determined by exogenous factors (e.g., the birth rate) and does not depend on the size of the capital stock. Hence, a contraction in capital intensity translates into a proportionate decrease in the real economy.

Scenario 1: Under the first scenario, it follows that this contraction in capital intensity translates into a decrease of 0.34% in the GVA of the real economy. So, if the GVA of the real economy were £1.2 trillion, then the change in the economy's output is as follows:

$$\Delta Y_{\dagger}^{*} = -0.34\% \times \text{£}1.2 \text{ trillion} \approx -\text{£}4.1 \text{ billion}$$

This is the drop in output that would occur if the UK lost all access to "precise calibrations". Moreover, this is an estimate of what would be put at risk without having calibrations that are traceable to standards maintained by the NMS labs.

Now, consider the situation in which users wanting "precise calibrations" are signposted to VSL in the Netherlands. In this situation the drop in GVA is smaller than under Scenario 1.

Scenario 2: Under Scenario 2, this means that a 0.09% drop in capital intensity translates into a drop of 0.09% in output from the real economy. Therefore, if the GVA of the real economy is £1.2 trillion, then the resulting contraction in the economy is given by:

$$\Delta Y_{\dagger}^{*} = -0.09\% \times \text{£}1.2 \text{ trillion} \approx -\text{£}1.1 \text{ billion}$$

Thus, the contraction in the economy amounts to about £1.1 billion in lost output. This is the drop in output that would occur if users could only access "precise calibrations" by going to a foreign NMI.

32 BENEFIT-COST ANALYSIS

Through considering the damage to conformance testing that would be caused by discontinuing the NMS programme, we can value the benefits coming from the national measurement infrastructure. That is, it gives us an estimate of what would be lost if it wasn't for the NMS labs using their capabilities to act as an anchor for traceable calibrations.

The preceding sections of this report provided two different estimates of what would happen to the accuracy of calibrations without the NMS programme. Firstly, there's an extreme scenario in which, without the NMS labs, the UK completely loses access to "precise calibrations". Secondly, there's a more realistic scenario in which the UK retains some access to "precise calibrations" through VSL in the Netherlands. Nonetheless, even in this second scenario, there is a contraction in the use of "precise calibrations" due to the extra costs and difficulties of sending instruments abroad for calibration. The scenarios give bounds for the damage to conformance testing that would be caused by discontinuing the NMS programme. In terms of the lost GVA, the first scenario gives an extreme upper bound for the overall value that would be put at risk by defunding the NMS labs. Whereas the second scenario gives a more realistic estimate based on what would surely be lost even if arrangements with foreign NMIs (e.g., VSL) meant that users retained some access to "precise calibrations".

So far, we have considered the cost of losing access to precises calibrations from the perspective of businesses in the economy. Specifically, we have only considered the private costs. However, to properly evaluate the NMS programme, we must also account for the public costs of sustaining the capabilities of the NMS labs. Hence, it is important to include the NMS funding that is used to top-up and maintain the scientific knowledge embodied in the capabilities of the NMS labs.

Lastly, it is important to recognise that this section considers both the marginal benefit of changes in funding, as well as the average benefit from maintaining the full programme at its existing steady state level. Specifically, much of the BCR analysis in this section yields an 'average' rate of return rather an a 'marginal' rate of return. However, towards the end of this section, we derive a useful formula for converting an average rate of return into a marginal rate of return.

32.1 THE PUBLIC COST OF SUSTAINING THE CAPABILITIES OF THE NMS LABS

During the period considered in this study, the NMS labs receive around £111 million in funding from government in each financial year. This funding can be broken down as follows:

- £76 million in NMS funding.
- £13 million in other public funding from other programmes (e.g., SPF). This includes around £2 million for collaborating with businesses on their innovation projects.
- £22 million in funding for estates (e.g., renovating labs) and subscriptions.

This funding amounts to £111 million, but because NPL pays government £17 million in rent each year for the Teddington site, the net funding is reduced to £94 million.

The £76 million of NMS funding splits into the following categories: research (15%); development (40%); maintenance (40%); and knowledge transfer (5%). Which means that the spend on each of these elements is as follows: £11.4 million for research; £30.4 million for development; £30.4 million for maintenance; and £3.8 million for knowledge transfer.

Most readers will understand what's meant by 'R&D' and 'KT', but further explanation of the concept of 'maintenance' is helpful: 'Maintenance' refers to the work needed to maintain and update the existing measurement capabilities of the NMS labs. This kind of work includes servicing the apparatus and facilities needed for the realisation of SI units (national standards). Maintenance also includes the work needed to retain UKAS accreditation, and the training of new staff so that they can deliver measurement services. In some areas, maintenance also include running proficiency testing schemes and the provision of certified reference materials. Lastly, although, the maintenance projects don't generate benefits in themselves, it is the labs' capabilities that makes all their other impact generating activities possible, and so these maintenance projects are vital to the programme.

The key question is how much of this funding is essential to sustain the capabilities of the NMS? The funding can be split into 'discretionary' and 'essential' components as follows:

- The 'discretionary' component is composed of funding from other programmes and the NMS funding allocated to research and knowledge transfer projects.
- The 'essential' component is composed of funding for estates and the NMS funding for maintenance and development projects.

Perhaps, from an outside perspective, it could have been argued that only the NMS funding spent on maintaining the existing capabilities is truly essential. But, without a stream of metrology-related innovations - flowing from the NMS's development projects - the capabilities of the NMS labs would soon cease to be leading edge. Hence, to keep the NMS labs at the frontier of measurement science, we really ought to include the cost of these development projects within the essential component of the NMS. In contrast, much of the research could probably be done either by foreign NMIs or at science departments within some of the UK's top universities. Moreover, in the short run, cutting the funding for research probably wouldn't materially affect the existing capabilities of the NMS labs. (There is still a danger that some of the more research minded scientists decide to seek research jobs outside of the NMS labs and take their expertise with them when they leave.)

Hence, an estimate of funding that is 'essential' to sustaining the capabilities of the NMS is given by:

$$\text{essential funding} = £22\text{ million} + £76\text{ million} \times (40\% + 40\%) \approx £83\text{ million} \qquad 32\text{-}1$$

For numerical convenience, let this estimate be rounded to £80 million. If we add this £80 million in essential public funding to the private costs from enhanced conformance testing, then we get the social cost of the extra spending on measurement science.

Lastly, our analysis of the costs shows that 75% of the total funding is essential to sustaining the flow of benefits that come purely from the measurement infrastructure. The remaining 25% of funding generates other kinds of benefit (e.g., enhanced innovation amongst a group of regularly supported businesses), which fall outside the scope of the model used in this report, and so additional models are needed to quantify such benefits. However, by its very nature, the 'essential' component of the funding is the foundation of all these other kinds of benefit. That is, the other kinds of benefits (e.g., enhanced innovation amongst a group of regularly supported businesses) could not exist without the funding that sustains the core infrastructure.[45]

32.2 UPPER BOUND FOR THE RATE OF RETURN

Let us consider the first scenario in which the NMS labs were defunded and there was no alternative source of "precise calibrations" for UK-based users who had previously relied on calibrations traceable to the NMS labs. No doubt, a work-around would emerge so that calibration labs could occasionally send their own instruments to foreign NMIs to be recalibrated, but the frequency of this would be just sufficient to backstop a decline in accuracy. This would ensure that they could offer "basic calibrations" to customers, but they would no longer be able to offer "precise calibrations" to their customers. This scenario gives an extreme upper bound for the benefit of the NMS programme.

The results for Scenario 1 are summarised in the following proposition:

Proposition 32-1: *If users had no access to "precise calibrations", then businesses would revert to using "basic calibrations" with a Test Accuracy Ratio (TAR) of 1:4. This loss of precision would cause a 3% increase in the Relative Standard Deviation (RSD) of measurements made in conformance testing. The consequences of this extra measurement uncertainty are as follows:*

1. *Assuming society won't compromise the statistical power of conformance tests, this increase in the RSD causes the likelihood of type-1 errors (false positives) to increase from 0.3% to 0.5%.*
2. *The rise in the RSD of the tests reduces the optimal inspection frequency by 4.2%, and so the spending on conformance testing decreases by 4.1%. Since the UK spends around £28.8 billion on conformance testing, this decrease amounts to a drop in spending of £1.2 billion.*
3. *This rise in the RSD lowers the Marginal Product of Capital (MPK) and causes the economy's capital intensity to decrease by 0.34%.*
4. *Lastly, for a real economy with a GVA of £1.2 trillion, this drop in capital intensity causes the economy's output to contact by £4.1 billion.*

So, without any access to "precise calibrations", conformance testing activities become somewhat less effective, prompting businesses cut back their spending on such activity. The amount spent by businesses on conformance testing would drop by £1.2 billion, however, there would also be a £4.1 billion decrease in the GVA of the real economy. So, the private net-benefit from users continuing to access to "precise calibrations" is given by:

[45] The various benefits are distinct and separate, although, they depend on the same set of capabilities. This means that to get an estimate of the overall benefit of the NMS, we need to add together the private net-benefits from each of the mechanisms.

Private Net-Benefit = £4.1 billion – £1.2 billion = £2.9 billion 32-2

The benefit-cost ratio for continued access to precise calibrations is given by:

$$\text{Private BCR} = \frac{\text{£4.1 billion}}{\text{£1.2 billion}} = 3.42$$

So, £1.00 spent on conformance testing by businesses yields a gross-benefit of £3.42, and a net-benefit of £2.42. This figure of £2.42 represents the private rate-of-return on the money spent by businesses due to enhanced conformance testing that is underpinned by the NMS labs. Furthermore, the private costs and the private benefits occur almost concurrently (i.e., within the same year).

Before moving on to consider the social return, let us find the private return on the public funding. Since public funding for the NMS was around £80 million in the period 2015 - 2019, the private return on the public funding is given by:

$$\frac{\text{Private Net Benefit}}{\text{Public Funding}} = \frac{\text{£2.9 billion}}{\text{£80 million}} = 36$$

So, £1 spent on the NMS by the government yields a private net-benefit of £36. Note that this is an average return rather than a marginal rate-of-return.

If we now add the £80 million of NMS funding to the £1.2 billion in private costs from enhanced conformance testing, then we get a social cost of £1.28 billion. Previously, we showed that without access to "precise calibrations", there would be a £4.1 billion drop in the GVA of the real economy. Thus, an extreme upper bound for the social net-benefit from the NMS programme is given by:

Social Net-Benefit = £4.1 billion – £1.28 billion = £2.82 billion 32-3

Moreover, based on our estimates for Scenario 1, an extreme upper bound for the BCR of the NMS programme becomes:

$$\text{Social BCR} = \frac{\text{£4.1 billion}}{\text{£1.28 billion}} = 3.20$$

So, on average, £1:00 spent on measurement science by society yields a gross-benefit of £3.20 and a net-benefit of £2.20. However, keep in mind that this figure of £2.20 represents an average rate-of-return, which is somewhat higher than the corresponding marginal rate-of-return.

Lastly, based on this benefit-cost analysis, an estimate of the social return on the public funding allocated to the NMS is given by:

$$\frac{\text{Social Net Benefit}}{\text{Public Funding}} = \frac{\text{£2.82 billion}}{\text{£80 million}} = 35$$

So, on average, £1 spent on the NMS by the government yields a social net-benefit of £35. This means, the social return on public funding is extremely large. However, this should not be interpreted as saying that doubling the £80m of public funding would yield this enormous net-benefit. Firstly, this is an average return rather than a marginal return. And, secondly, this represents the extreme case where the NMS is defunded without lining up a foreign NMI to act as an alternative source of "precise calibrations". So, this is the value of what would be lost from the economy if the NMS labs were defunded (in the extreme case where there isn't an arrangement with a foreign NMI) relative to the savings that could be achieved by cutting the NMS programme in its entirety.

32.3 LOWER BOUND FOR THE RATE OF RETURN

Let us consider the second scenario in which the NMS labs were again defunded, but this time users wanting “precise calibrations” were signposted to VSL in the Netherlands. Compared to Scenario 1, this second scenario provides a more realistic lower bound for the benefit of the NMS programme.

Proposition 32-2: *If users had to go to VSL for “precise calibrations”, then there would be a 31% decrease in the UK’s use of “precise calibrations”. Hence, around one-third of the businesses who currently benefit from “precise calibrations” would revert to using “basic calibrations”. The loss of precision amongst such businesses would cause a 1% increase in the Relative Standard Deviation (RSD) of measurements made during conformance testing. The consequences of this extra measurement uncertainty are as follows:*

1. *Assuming society won’t compromise the statistical power of conformance tests, this increase in the RSD causes the likelihood of type-1 errors (false positives) to increase from 0.30% to 0.35%.*
2. *The rise in the RSD reduces the optimal inspection frequency by 1.1%, meaning that spending on conformance testing also decreases by 1.1%. Since the UK spends around £28.8 billion on conformance testing, this decrease amounts to a drop in spending of £317 million.*
3. *This rise in the RSD lowers the Marginal Product of Capital (MPK) and causes the economy’s capital intensity to decrease by 0.09%.*
4. *Lastly, for a real economy with a GVA of £1.2 trillion, this drop in capital intensity causes the economy’s output to contact by £1.1 billion.*

If users had to go to VSL for “precise calibrations”, then the spending on conformance testing would drop by about £317 million, however, the decrease in the economy’s GVA would be about £1.1 billion. So, the private net-benefit from users going to the NMS labs for “precise calibrations” (as opposed to them having to travel to VSL) is given by:

$$\text{Private Net-Benefit} = £1.1 \text{ billion} - £317 \text{ million} = £783 \text{ million} \tag{32-4}$$

The benefit-cost ratio for continued access to “precise calibrations” is given by:

$$\text{Private BCR} = \frac{£1.1 \text{ billion}}{£317 \text{ million}} = 3.47$$

So, spending £1.00 on conformance testing yields a benefit of £3.47 and a net-benefit of £2.47. This figure of £2.47 represents the private rate-of-return on the money spent by businesses on enhanced conformance testing. Also, note that these costs and benefits occur concurrently (same year).

Before moving on to consider the social return, let us find the private return on the public funding. Since public funding for the NMS was around £80 million in the period 2015 - 2019, the private return on the public funding is given by:

$$\frac{\text{Private Net Benefit}}{\text{Public Funding}} = \frac{£783 \text{ million}}{£80 \text{ million}} = 9.79$$

So, £1:00 spent on the NMS by the government yields a private net-benefit of £9.79. Note that this is an average return rather than a marginal rate-of-return.

If we now add the £80 million of NMS funding to the £317 million in private costs from measurement activity, then we get a social cost of £397 million. Previously, we showed that if users had to go to VSL for “precise calibrations”, there would be a £1.1 billion decrease in the economy’s GVA. Thus, a lower bound for the social net-benefit from the NMS programme is given by:

$$\text{Social Net Benefit} = £1.1 \text{ billion} - £397 \text{ million} = £703 \text{ million} \quad (32\text{-}5)$$

Moreover, based on Scenario 2, an upper bound for the social BCR becomes:

$$\text{Social BCR} = \frac{£1.1 \text{ billion}}{£397 \text{ million}} = 2.77$$

So, £1:00 spent on measurement science by society yields a benefit of £2.77, and a social net-benefit of £1.77. However, keep in mind that £1.77 represents an average rate-of-return, which is somewhat higher than the corresponding marginal rate-of-return.

Finally, the social return on the public funding is given by:

$$\frac{\text{Social Net Benefit}}{\text{Public Funding}} = \frac{£703 \text{ million}}{£80 \text{ million}} = 8.79$$

So, £1:00 spent on the NMS by the government yields a social net-benefit of £8.79. Note that this is an average return rather than a marginal rate-of-return.

32.4 MARGINAL RATE OF RETURN

What about a situation in which the NMS labs keep all the maintenance funding but there is a cut to some of the funding for development? In other words, what would be the impact of a cut in NMS development projects? To answer this question, we need an estimate of the marginal rate of return on public funding. This section shows that the marginal rate of return is proportional to the average rate of return but, necessarily, smaller because of the social rate of time preference (SRTP).

Let $\Pi(t)$ denote the private net-benefits attributable to the NMS at time t. In other words, $\Pi(t)$ is the private benefit minus the private cost. Secondly, suppose that there's a stock of knowledge embodied in the capabilities of the NMS; and let $\mathbb{K}(t)$ denote the size of this knowledge stock at time t in terms of the accumulated R&D spending. Next, suppose that the private net-benefits coming from the NMS is proportional to the knowledge stock, meaning that: $\Pi(t) \propto \mathbb{K}(t)$. For example, if the stock of knowledge dropped by 10%, then the private net-benefits (as detailed above) would also drop by 10%. Moreover, for now, we'll suppose that a change in the size of the knowledge stock has a concurrent effect on the flow of benefits; but towards the end of this section, we will introduce a six-year delay before a public investment in R&D translates into a flow of economic benefits.

Unlike much of the basic scientific research that is done in universities, the R&D projects of the NMS constitute a form of applied research. Consequently, the relevance and importance of this kind of knowledge declines as the economy and its technology continue to change and evolve. Hence, we assume that the stock of knowledge has been built up by the NMS programme investing a certain fixed amount each year in the capabilities of its labs, but that the labs' existing knowledge depreciates at a rate of 15% each year.[46] Moreover, suppose that up until the present time, the NMS programme has always supplied £80 million in funding each year but that now there's a sudden jump to a different level of funding. So, let the t = 0 denote the present; t < 0 denote the past; and t > 0 denote the future. The annual public investment going into the NMS labs from government can be represented as follows:

$$\mathbb{I}(t) = \begin{cases} \mathcal{I}_0 \text{ if } t \leq 0 \\ \mathcal{I}_+ \text{ if } t > 0 \end{cases} \quad (32\text{-}6)$$

where $\mathcal{I}_0$ and $\mathcal{I}_+$ are parameters representing current and future public funding, respectively. Suppose that $\mathcal{I}_0 = £80$ million, and that $\mathcal{I}_+ \neq \mathcal{I}_0$, which means there is a change in public funding: $\Delta\mathcal{I} \equiv \mathcal{I}_+ - \mathcal{I}_0$.

[46] This depreciation rate is based on estimates from the ONS.

Following the same kind of logic as used for the evolution of the capital stock, the differential equation for the evolution of the knowledge stock is as follows:

$$\frac{d\mathbb{K}(t)}{dt} = \mathbb{I}(t) - 15\%\mathbb{K}(t) \qquad 32\text{-}7$$

So, for $t \leq 0$, this differential equation becomes: $\dot{\mathbb{K}}(t) = \mathcal{I}_0 - 15\%\mathbb{K}(t)$. And, for $t > 0$, it becomes: $\dot{\mathbb{K}}(t) = \mathcal{I}_+ - 15\%\mathbb{K}(t)$, where $\dot{\mathbb{K}} \equiv d\mathbb{K}/dt$.

Up until now, the yearly funding has always been $\mathcal{I}_0$, and so the stock will have settled into a steady state. That is, $\dot{\mathbb{K}}(0) = 0$, which implies that: $\mathcal{I}_0 - 15\%\mathbb{K}(0) = 0$. Consequently, the size of the stock of knowledge, in this steady state, is given by:

$$\mathbb{K}(0) = \mathcal{K}_0 = \frac{\mathcal{I}_0}{15\%} \qquad 32\text{-}8$$

Furthermore, given that $\mathcal{I}_0 = \pounds 80 \text{ million}$, it follows that:

$$\mathcal{K}_0 = \frac{\pounds 80 \text{ million}}{15\%} = \pounds 533 \text{ million}$$

This figure represents the baseline value of the existing knowledge stock. Hence, we can use this result to "initialise" the stock of knowledge, thereby giving us one of the boundary conditions needed to solve the differential equation for $t > 0$. And, according to which, its solution is as follows:

$$\mathbb{K}(t) = \frac{\mathcal{I}_+}{15\%} - \left(\frac{\mathcal{I}_+}{15\%} - \mathcal{K}_0\right)\exp(-15\%t) \qquad 32\text{-}9$$

Recall that $\mathcal{K}_0 = \mathcal{I}_0/15\%$, and so if $\mathcal{I}_+ = \mathcal{I}_0$, then we get back $\mathbb{K}(t) = \mathcal{K}_0$, as we must.

This equation for $\mathbb{K}(t)$ leads to the following expression for the proportional change in the size of the knowledge stock relative to its baseline:

$$\frac{\mathbb{K}(t) - \mathcal{K}_0}{\mathcal{K}_0} = \left(\frac{\mathcal{I}_+ - \mathcal{I}_0}{\mathcal{I}_0}\right)[1 - \exp(-15\%t)] \qquad 32\text{-}10$$

Recall that the private net-benefit at time t will be proportional to $\mathbb{K}(t)$, which gives us the following identity:

$$\frac{\Pi(t) - \pi_0}{\pi_0} = \frac{\mathbb{K}(t) - \mathcal{K}_0}{\mathcal{K}_0} \qquad 32\text{-}11$$

where π_0 denotes the baseline for private net-benefits attributable to the NMS labs. From this, it follows that the change in the private net-benefit at time t (expressed as an increment from the baseline at t = 0) can be written as:

$$\Pi(t) - \pi_0 = \pi_0 \times \left(\frac{\mathbb{K}(t) - \mathcal{K}_0}{\mathcal{K}_0}\right) \qquad 32\text{-}12$$

In the first scenario, $\pi_0 = \pounds 2.9 \text{ billion}$, whereas, in the second scenario, $\pi_0 = \pounds 783 \text{ million}$. However, in both scenarios, $\mathcal{I}_0 = \pounds 80 \text{ million}$ and $\mathcal{K}_0 = \pounds 533 \text{ million}$. So, in the first scenario, the average private return on public funding is: $\pi_0/\ \mathcal{I}_0 = 36$. Whereas, in the second scenario, the average private return on public funding is: $\pi_0/\ \mathcal{I}_0 = 9.79$.

The idea is that changing the level of funding alters the private net-benefits attributable to the NMS. However, finding the marginal return involves comparing the change in the private net-benefits ($\Pi(t) - \pi_0$) to the change in the public funding ($\mathcal{I}_+ - \mathcal{I}_0$). HMT's Green Book gives us 3.5% as the discount rate reflecting the social rate of time preference (SRTP). From which, it follows that the discount factor is: $\exp(-3.5\%t)$. The present value (PV) of the change in funding and the PV for the change in private net-benefits is found by multiplying each of them by the discount factor and integrating from t = 0 to $t = \infty$.

Applying a discount rate of 3.5% to the change in funding and the change in private net-benefits yields the following formulae for the corresponding PVs:

$$\text{PV of Change in Funding} = \int_0^{\infty} (\mathcal{I}_+ - \mathcal{I}_0).\exp(-3.5\%t)dt \qquad 32\text{-}13$$

$$\text{PV of Change in Private Net Benefits} = \int_0^{\infty} [\Pi(t) - \pi_0].\exp(-3.5\%t)dt \qquad 32\text{-}14$$

The marginal private return on the public funding must be given by the following formula:

$$\text{Marginal Private Return on Funding} = \frac{\text{PV of Change in Private Net Benefits}}{\text{PV of Change in Funding}}$$

Which using the formulae for the PVs then implies that:

$$\text{Marginal Private Return on Funding} = \frac{\int_0^{\infty} [\Pi(t) - \pi_0].\exp(-3.5\%t)dt}{\int_0^{\infty} (\mathcal{I}_+ - \mathcal{I}_0).\exp(-3.5\%t)dt} \qquad 32\text{-}15$$

This leads to the following proposition:

Proposition 32-3: Suppose that the annual flow of public funding for the NMS labs has remained about the same for a long time and the stock of knowledge embodied in their capabilities has reached a steady state. Let $\boldsymbol{\mathcal{I}_0}$ denote the current (and historic) level of public funding for the NMS labs. This stock of knowledge sustains an annual flow of private net-benefits; and let $\boldsymbol{\pi_0}$ denote the private net-benefit attributable to the NMS labs. Lastly, suppose that a change in the size of knowledge stock has an immediate effect on the flow of economic benefits.

1. Now, suppose that the annual funding shifts from $\boldsymbol{\mathcal{I}_0}$ to $\boldsymbol{\mathcal{I}_+}$. The marginal private return on public funding is found by dividing the present value (PV) of the change in the private net-benefits by the present value (PV) of the change in the public funding.
2. If the stock of knowledge depreciates at a rate of 15% per year and the yearly discount rate is 3.5%, then the marginal private return on public funding is given by:

$$\frac{\text{PV of Change in Private Net Benefits}}{\text{PV of Change in Funding}} = \frac{\pi_0}{\mathcal{I}_0} \times \left(\frac{15\%}{15\% + 3.5\%}\right) = \frac{\pi_0}{\mathcal{I}_0} \times 81\% \qquad 32\text{-}16$$

   Where $\pi_0/\mathcal{I}_0$ is the average private return on public funding in the existing steady state.
3. The marginal return is the average return multiplied by the factor in brackets. Since the factor in brackets is less than one, the marginal return is necessarily less than the average return.

The proof of this proposition runs as follows:

Proof. The marginal private return on public funding is given by the following formula:

$$\frac{\text{PV of Change in Private Net Benefits}}{\text{PV of Change in Funding}} = \frac{\int_0^{\infty} [\Pi(t) - \pi_0].\exp(-3.5\%t)dt}{\int_0^{\infty} (\mathcal{I}_+ - \mathcal{I}_0).\exp(-3.5\%t)dt}$$

By substituting for $\Pi(t) - \pi_0$ using an earlier expression, we get:

$$\frac{\text{PV of Change in Private Net Benefits}}{\text{PV of Change in Funding}} = \frac{\int_0^\infty \pi_0 \left(\frac{\mathbb{K}(t) - \mathcal{K}_0}{\mathcal{K}_0}\right).\exp(-3.5\%t)dt}{\int_0^\infty (\mathcal{J}_+ - \mathcal{J}_0).\exp(-3.5\%t)dt}$$

Substituting for the proportional change in the knowledge stock, yields:

$$\frac{\text{PV of Change in Private Net Benefits}}{\text{PV of Change in Funding}} = \frac{\int_0^\infty \pi_0 \left(\frac{\mathcal{J}_+ - \mathcal{J}_0}{\mathcal{J}_0}\right)[1 - \exp(-15\%t)].\exp(-3.5\%t)dt}{\int_0^\infty (\mathcal{J}_+ - \mathcal{J}_0).\exp(-3.5\%t)dt}$$

Some simplification, along with the cancelling of common factors, gives:

$$\frac{\text{PV of Change in Private Net Benefits}}{\text{PV of Change in Funding}} = \left(\frac{\pi_0}{\mathcal{J}_0}\right)\frac{\int_0^\infty [\exp(-3.5\%t) - \exp(-18.5\%t)]dt}{\int_0^\infty \exp(-3.5\%t)dt}$$

Computing the integrals, yields:

$$\frac{\text{PV of Change in Private Net Benefits}}{\text{PV of Change in Funding}} = \left(\frac{\pi_0}{\mathcal{J}_0}\right)\frac{[(1/3.5\%) - (1/18.5\%)]}{(1/3.5\%)}$$

A little rearrangement completes the proof.■

So far, we've assumed that a change in the knowledge stock has an immediate effect on the flow of economic benefits. However, according to a recent report by Frontier Economics[47], there's actually a delay of about six years before investments in public R&D start to generate their economic benefits. Hence, the benefits should really be further discounted to reflect the time lag between the public funding and the economic benefits. The appropriate discount factor is $\exp(-3.5\% \times 6) = 81\%$; and so, with a six-year lag between public spending and the arrival of the economic benefits, the PV for the change in private net-benefits becomes:

$$\text{PV of Change in Private Net Benefits} = \exp(-3.5\% \times 6)\int_0^\infty [\Pi(t) - \pi_0].\exp(-3.5\%t)dt$$

In contrast, this delay in the net-benefits has no effect on the PV of the funding, which leads to the following proposition:

Proposition 32-4: If there's a six-year delay between public investment in R&D and the corresponding economic benefits, then the marginal private return on public funding becomes:

$$\frac{\text{PV of Change in Private Net Benefits}}{\text{PV of Change in Funding}} = \frac{\pi_0}{\mathcal{J}_0} \times \left(\frac{15\%}{15\% + 3.5\%}\right) \times \exp(-3.5\% \times 6) \qquad 32\text{-}17$$

From this proposition, it follows that the marginal return is about 66% of the corresponding average return.

- In the first scenario, the average private return on public funding is £36 per £1 of public funding. (The calculation is: £2.9 billion / £80 million = 36.)
- In the second scenario, the average private return on public funding is £9.79 per £1.00 of public funding. (The calculation is: £783 million / £80 million = 9.79.)

So, according to the last proposition, the average private return of 36, becomes a marginal private return of 24. Similarly, for the second scenario, the average private return of 9.79, becomes a marginal private return of 6.46. Lastly, these are marginal private returns on public funding, which means the corresponding marginal social returns can be found by simply subtracting 1. Thus, for the first scenario, the marginal social return becomes 23. Whereas, for the second scenario, the marginal social return becomes 5.46.

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[47] Frontier Economics (2024), Returns to Public R&D. Report for the Department for Science, Innovation and Technology (DSIT). 12 December 2024.

32.5 A SUMMARY OF RESULTS

The NMS labs maintain and update the primary standards that underpin a distributed system for the certification of calibrations, and for ensuring comparability to corresponding standards around the world. Calibrations traceable to these primary standards are delivered to more than 74,000 businesses via a network of calibration laboratories distributed across the UK. (Since a top-tier calibration lab can supply calibration services to a second-tier calibration lab, this estimate of 74,000 businesses represents just the first tier of fanout across the economy.)

The benefit-cost analysis in this section shows that cuts in funding lead to a loss of economic benefits. We use the lower bound (Scenario 2) as the most reasonable estimate of what would be lost without the NMS labs. The upper bound (Scenario 1) gives an estimate of what would be put at risk without the NMS labs. So, the lower bound is what the UK would surely lose, and the upper bound for the value of is what would be put in jeopardy.

Before outlining the impact of the NMS, a few key things must be noted. Firstly, a marginal cut to the NMS would be less significant per pound of saving than if the programme were scrapped in its entirety. In essence, a marginal cut to benefits is less than an average cut in benefits. Secondly, if funding for the UK's measurement infrastructure was cut in its entirety, then the programme's other benefits mechanisms (research, innovation, knowledge transfer) would also cease to operate. Finally, there are two distinct kinds of loss depending on the scale of the cut:

- If the NMS was cut in its entirety, the average return on public funding should be used to get an estimate of the economic damage.
- If the NMS lost a proportion of its funding but continued as a programme, then the marginal return on public funding should be used to get an estimate of economic losses.

From this economic analysis, if the UK stopped funding the NMS labs but came to an arrangement with a foreign NMI (such as, VSL in the Netherlands), we would see an average social loss to the economy of £8.79 for each £1.00 saved by the government due to no longer funding the NMS. The argument can be summarised as follows:

- "Basic calibrations" have a Test Accuracy Ratio (TAR) of around 1:4. In contrast, using "precise calibrations" reduces the Relative Standard Deviation (RSD) of measurements but about 3%.
- Amongst the many thousands of businesses who directly (or indirectly) depended on the NMS labs, the extra cost of having to go abroad to access to "precise calibrations" leads to a fall of 31% in the use of top-quality calibrations that are traceable to highly accurate national standards. (This is an extrapolation based on an econometric study which found that a customer's distance from NPL negatively effects how regularly they use its services.)
- A decline in the use of "precise calibrations" across the economy, reduces the effectiveness of the businesses' conformance testing activities, as measurements become somewhat less reliable. In other words, businesses would experience a drop in the rate-of-return on their conformance testing activities, due to a higher rate of "false positives" and an increase in the scrap rate. Consequently, we find that the amount spent by businesses on conformance testing would fall by 1.1% in response to its reduced effectiveness. Given that the UK currently spends around £28 billion on conformance testing, this drop in their spending yields a notional saving of £317 million for businesses.
- However, along with this drop in the amount that businesses spend on conformance testing there also comes a corresponding drop in the marginal product of capital,

which then leads to a 0.1% decrease in the economy's capital intensity (given that the cost-of-capital in equilibrium is fixed by parameters that remain almost unchanged). We find that the consequent drop in equilibrium labour productivity results in a loss to the economy of £1.1 billion in GVA. (The proportionality of output per worker and capital intensity comes from the circular flow equation connecting savings and investment.) Lastly, since the government would save £80 million from scrapping the NMS, the end result is a net economic loss to the UK of £700 million in GVA.

Of more relevance to considerations during a government spending review is a cut to some portion of the NMS funding. In other words, the NMS might one day be earmarked for a cut in funding, necessitating its withdrawal from certain areas of measurement activity in proportion to the scale of the cut.

For context, the NMS currently covers about 75% of the core measurement capabilities as outlined by BIPM's database (of CMCs). The breadth of coverage would necessarily reduce in proportion to the cut in funding; meaning that if the NMS labs scaled back their offering, then businesses requiring high accuracy calibrations in the "mothballed" areas would have to send their instruments to a foreign NMI. The analysis in this report shows that this would lead to a marginal social loss to the economy of £5.46 per £1 saved through cuts to the programme. This estimate includes a discount factor for a presumed 6-year delay in the effects being felt in the economy. This is due to the cuts falling upon the development of the measurement infrastructure - not the maintenance projects, which would still be safeguarded. (If you cut maintenance, you default to the large losses from scrapping the whole NMS.)

This concludes Part 5 of the report. The final sections summarise the main findings and make suggestions for further work.

33 FURTHER WORK

33.1 UPDATING THE PARAMETERS IN THE MODEL

The analysis developed in this document will really come to life once we have updated estimates of the parameters using data from a forthcoming measurement survey. Hence, an empirically focussed follow-on study is required to update the parameters in the model. Moreover, there is strong potential for this model to be used in combination with sector-level case studies. By this means, it is hoped that future studies will be able to make reasonable estimates of a sector's spending on conformance testing and then use the model to infer the sector-level benefits.

33.2 STEADY STATE INVESTMENT IN THE INFRA-TECHNOLOGY

Further work is needed to better model the connection between investments in the infra-technology and the pace at which engineers can do their conformance testing work. Firstly, a significant fraction of technical standards is associated with measurement activities or product verification. Secondly, standards documents can be thought of as a stock of practical knowledge, detailing the best process or procedure for achieving a specified type of output. This freely available knowledge constitutes economy's infra-technology, and improvements in this knowledge feeds through to productivity growth. Finally, to maintain its relevance the stock of standards will need to be updated at about the same rate as the capital stock is being refreshed (with old items leaving and new items entering). To better understand the effect of public investment in infra-technologies, a model is needed that integrates these three elements.

33.3 A MODEL FOR RESILIENCE AND VULNERABILITY

As an extension, it might be possible to recast the model so that it could be used to analyse the economic impact of changes to the resilience of the production process. The economic effects of such changes could be modelled by varying the transition rate, ε, as this parameter sets the vulnerability of production processes to the onset of malfunctions. For example, if new technologies were to make production processes more vulnerable to hacking or sabotage, then ε would increase, whilst investing in mitigations would lead to a decrease in ε.

33.4 THE RAMSEY MODEL

Alongside empirical studies to estimate the parameters, there's potential for a further theoretical development of the model itself through building on a macroeconomic approach (dynamic optimisation of a social welfare function) that was pioneered by Frank Ramsey.

In the 1920s, Frank Ramsey created a framework that strengthened the microeconomic foundations of macroeconomic models. This advance was achieved by introducing a representative citizen and a social planner who runs the economy in a way that maximises the welfare of this representative citizen. This social welfare function corresponds to the integral of instantaneous utility, over a long period of time, during which the planner seeks to maximise a measure of aggregate happiness. That is, Ramsey took a model much like the original Solow model (for production and capital accumulation) but then added the following elements:

- A social welfare function based on the utility of a representative citizen. (Aggregate utility in an economy populated by identical citizens can be stated in terms of the happiness of a single infinitely lived agent.)
- A benevolent social planner who controls the economy and whose objective is to maximise social welfare.
- A Hamiltonian for the social planner's dynamic optimisation problem, from which is obtained "equations of motion" for the economy, in the form of a system of differential equations.

Behind this mathematical set-up is an optimistic vision of government working in tandem with the "invisible hand" of the market to produce the best possible outcome for citizens. Using the device of a social planner who maximises social welfare, Ramsey derived "equations of motion" for the economy, and then solved these differential equations to find the optimal path. In Ramsey's case, dynamic optimisation was used to endogenise the way that output is split between consumption and saving, leading to the Keynes-Ramsey rule for the evolution of consumption, which is an improvement on the exogenous savings rate used in the Solow model.

Just as the Ramsey model endogenised the savings rate, an extended Ramsey model would fully endogenise the frequency of inspections found in our extended Solow model. It's reasonable to suppose that the results of the classic Ramsey model (e.g., "equations of motion") will only be slightly perturbed by introducing new elements to the model. Thus, it's also reasonable to look for solutions in the vicinity of solutions to the classic Ramsey model. However, the adapted model would be significantly more complicated to analyse than the classic Ramsey model: Whereas the classic model had one state variable (capital) and one control variable (consumption), the new model would have two control variables (frequency of inspections and consumption) and two state variables (reliability of production and capital intensity).

An extended Ramsey model could be used to explore the optimal balance between (1) investment in capital (e.g., plant and machinery), and (2) buying-in engineers to oversee production so that malfunctioning machines are detected quickly. Moreover, this optimal balance will depend on the quality of the economy's infra-technology. Capital deepening directly raises labour productivity, whereas the expertise of the engineers helps to maintain the reliability of production by finding, and resetting, malfunctioning machines. Both (1) and (2) can raise per capita output but, as output is finite in the short term, more spending on one mechanism comes at the expense of less spending on the other mechanism. Hence, a future study should explore society's relative spending on these two mechanisms for raising per capita output, which is itself influenced by the effectiveness of the national quality infrastructure.

34 CONCLUSION

This section concludes by summarising the main findings of this study.

34.1 THE STEADY-STATE EQUILIBRIUM

The model developed in this study yields a system of differential equations: one for the evolution of the capital intensity and another for the reliability of production. A two-dimensional phase diagram, whose axes represent the economy's capital intensity and the reliability of production, can help picture the dynamics of this system of differential equations. Moreover, this phase diagram provides a means of finding an equilibrium in which both variables remain constant (at their steady-state values).

The formulae for the steady-state values of these variables provide the mathematics behind a theory-of-change, which explains how changes in the basic parameters of the model affect economic outcomes:

- Labour productivity is positively affected by the reliability of production in the steady state; meaning that it rises if engineers get better at finding the malfunctions. Moreover, the efficiency of production depends on the "regret rate" of the conformance tests, which refers to the rate at which perfectly viable output is mistakenly scrapped.
- The equilibrium level of the economy's capital intensity increases, when the efficiency of the production process rises. This causes a rise in per capita income, meaning that citizens become more prosperous. Furthermore, with a fixed savings rate, a rise in per capita income leads to a bigger pool of savings, which can then be used by businesses to fund their investments in new capital equipment.

Therefore, capital intensity indirectly depends on the "regret rate" through its connection to the efficiency of production and the prosperity of citizens.

Next, in the short term, one would expect the rental rate (the marginal product of capital) and capital intensity move in opposite directions. However, in the long run, the positive effect (on the marginal product of capital) from an improvement in the efficiency of production processes almost exactly offsets the negative effect on the rental rate from an increase in capital intensity (through an increase in the supply of capital items). It follows that the equilibrium rental rate will hardly change even when engineers get better at finding and fixing the malfunctioning machines. In other words, an improvement in the efficiency of production increases the demand for capital but, in equilibrium, the price of capital equipment remains almost unchanged. (Such results are one of the benefits of using general equilibrium models rather than partial equilibrium models.)

Lastly, in the steady state, the net rental rate (rents minus the cost of engineers) is proportional to the gross investment rate. This is a version of Piketty's famous formula, as espoused in his book: '*Capital in the Twenty-First Century*.' Next, it can be shown that the economy's net revenue (revenue minus the cost of engineers) is proportional to the level of gross investment. So, in equilibrium, society's per capita consumption ("prosperity") is an increasing function of the economy's capital intensity. Hence, anything that increases the capital intensity (such as, engineers getting better at finding genuine malfunctions) also increases peoples' living standards, which feeds back on capital intensity by increasing the flow of savings used for investment.

34.2 A NUMERICAL METHOD FOR ESTIMATING THE PARAMETERS

The scrap rate, rebate rate, and the "portion-size" are known quantities, that have been estimated using data on the economy. In particular, the values of the known parameters are as follows:

- Scrap Rate = 3.7%
- Rebate Rate = 1.5%
- Success Rate = 94.8%
- Gross Investment Rate = 6.3%.
- Portion-Size = £5.1 million.
- Engineer's Wage = £33 thousand
- Spending on CT as a Percentage of GVA = 2%

These estimates relate to the period 2015 to 2019 and will be updated using data from a forthcoming NMS survey.

However, some of the parameters in our model are much more difficult to determine from data. And, so to find such parameters, the model was "operationalised" by selecting values for the unknown parameters that yield results consistent with values of the known parameters. Such an approach was possible because the analysis in this study had already yielded formulae for the equilibrium values of the variables, along with a first-order condition for the optimal frequency of inspections. Moreover, the model contains the same number of equations as unknown parameters. (Seven equations and seven unknowns.) Hence, this set of simultaneous equations were used to infer the values of unknown parameters. A numerical analysis gave the following results:

- The reliability of the production process is $v^* = 95.1\%$.
- The regret rate is $\theta = 0.3\%$.
- The detection rate is $\phi = 69.4\%$.
- The transition rate is $\varepsilon = 3.9\%$.
- The frequency of inspections is $n = 1.18$.
- The likelihood of type-1 errors is $p_{1|0} = 0.3\%$.
- An engineer's span of control (pace of testing) is $a = £6.0$ million.

These are reasonable extrapolations based on what's known about the model's observable parameters.

34.3 HEADLINE RESULTS

The quality of the infra-technology that provides the technical basis for standards impacts the pace at which an engineer can inspect the machines under their supervision. Much of this infra-technology belongs to the science of metrology, which constitutes a kind of public good that is developed and maintained by the specialist laboratories funded through the National Measurement System (NMS) programme.

This economic analysis shows that if the UK stopped funding the NMS labs but came to an arrangement with a foreign NMI (such as, VSL in the Netherlands), we would see an average social loss to the economy of £8.79 for each £1.00 saved by the government due to no longer funding the NMS.

Perhaps, of more relevant to a government spending review, the NMS could experience a cut in funding, forcing it withdraw from certain areas of measurement in proportion to the cut in funding. For context, the NMS currently covers about 75% of the Core Measurement Capabilities (CMCs) as outlined by BIPM's database. If the NMS labs scaled back their offering, then the businesses needing high accuracy calibrations in the "mothballed" areas would have to send their instruments to a foreign NMI. The analysis in this report shows that this would lead to a marginal social loss to the economy of £5.46 per £1 saved through cuts to the programme.

Lastly, the analysis developed in this study shows how the model's parameters determine the behaviour of the system. Establishing updated values for these parameters is outside the scope of this study but will become the subject of further work. Nonetheless, the ability to identify the key parameters, and then combine them in a consistent model, is a crucial step towards a full quantification of the costs and benefits. Such parameters may ultimately form the basis for metrics that will enable us to track changes in the performance of the system.

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