LETTER TO THE EDITOR • OPEN ACCESS

Relating SI units to the 'defining constants'

OPEN ACCESS
IOP Publishing | Bureau International des Poids et Mesures
Metrologia
Metrologia 61 (2024) 043001 (4pp)
https://doi.org/10.1088/1681-7575/ad4cef

Letter to the Editor

Relating SI units to the 'defining constants'

Peter Blattner[1,*] and **Richard J C Brown**[2,3]

[1] Federal Institute of Metrology (METAS), Lindenweg 50, 3003 Bern-Wabern, Switzerland
[2] National Physical Laboratory, Hampton Road, Teddington TW11 0LW, United Kingdom
[3] University of Surrey, Guildford GU2 7XH, United Kingdom

E-mail: peter.blattner@metas.ch

Received 17 April 2024, revised 15 May 2024
Accepted for publication 17 May 2024
Published 28 May 2024

Abstract

This contribution discusses the direct relationship of any SI unit to the defining constants. For many of the SI derived units the relationships to the defining constants are simpler than to the SI base units. Furthermore, some of the relationships directly reflect the fundamental quantum-physical effects that are exploited in the primary realisation of units. It is not the intention of this article to question the status of the seven SI base units and the distinction between the SI base units and the SI derived units, which have evolved historically and persist for pedagogical reasons, but instead to provide an alternative view of the SI. This alternative view also allows a broader and more modern presentation of metrology in general and demonstrates that, in respect of the ability to be directly realised from the defining constants, there is little practical difference between the SI base units and SI derived units.

Keywords: SI, quantities, units, dimensions, defining constants, metrology

1. Introduction

The birth of metrology was tied to the establishment of reliable reference measurement standards that instilled trust in trade and commerce. The Metre Convention of 1875 aimed to 'assure the international unification and improvement of the metric system' and initially involved the realisation, conservation and comparison of specific international physical prototypes for certain quantities. Subsequently, the International System of Units (SI) evolved and was structured conceptually: first through the introduction of different types of units [1] (i.e. base units, derived units and (temporarily) supplementary units), and later through reference to defining constants [2]. In the present definition of the SI the concept of these 'traditional' base units is still retained and promoted [3]. It is well recognized that the present SI has defined more base units than the minimum number necessary for describing physical effects. This is because the SI is a practical unit system intended to facilitate the measurements required to support trade and societal needs, and not only fundamental science.

The distinction between the SI base units and SI derived units has been recently extensively discussed [4]. The paper concludes that it 'seems sensible to retain the distinction between SI base units and SI derived units into the future'.

It is not the intention of this article to question the seven SI base units and the distinction between the SI base units and the SI derived units, which have evolved historically and persist for pedagogical reasons, but to provide an alternative view of the SI. This alternative view also allows a broader and more modern presentation of metrology in general.

In the revised SI any linearly independent set of units or combinations of units could serve as a basis set for unit analysis. An obvious choice are the units of the defining constants.

* Author to whom any correspondence should be addressed.

Using this approach the quantum-physical laws that were decisive in the revision of the SI can be neatly presented, and all SI units are given the same 'political' weight and scientific profile. The possibility of relating any SI unit directly to the defining constants is already mentioned in the introduction of chapter 2.3 of the SI-Brochure [3]: 'Prior to the definitions adopted in 2018, the SI was defined through seven base units from which the derived units were constructed as products of powers of the base units. Defining the SI by fixing the numerical values of seven defining constants has the effect that this distinction is, in principle, not needed, since all units, base as well as derived units, may be constructed directly from the defining constants.' However, this useful option is often forgotten, especially for the SI derived units, even though fundamental dependencies can be simply and elegantly illustrated. This also demonstrates that, in respect of the ability to be directly realised from the defining constants, there is little practical difference between the SI base units and the SI derived units.

2. Relating units to the defining constant

The core of the definition of today's SI are the seven defining constants, i.e. the unperturbed ground-state hyperfine transition frequency of the caesium 133 atom, $\Delta\nu_{\mathrm{cs}}$; the speed of light in vacuum, c; the Planck constant, h; the elementary charge, e; the Boltzmann constant, k; the Avogadro constant, N_{A}; and the luminous efficacy of monochromatic radiation of frequency 540×10^{12} Hz, K_{cd}.

The relationship of the SI base units to the defining constants has been discussed previously [5, 6] and is implicitly also stated in chapter 2.3.1 of the SI Brochure [3] where units are expressed by the product of a numerical value and one or more of the defining constants. In table 1 these relationships are expressed in terms of the units of the defining constants.

As the SI derived units can be expressed in terms of SI base units, and the SI base units can be expressed in terms of the units of the defining constants (see table 1) it is therefore also possible to relate the SI derived units directly to the units of the defining constants. Some of the relations are directly visible from the definition of the defining constant, in particular the derived unit hertz (Hz) is the same unit as the hyperfine transition frequency of the caesium 133 atom ($[\Delta\nu_{\mathrm{Cs}}]$), and the coulomb (C) is the same unit as the elementary charge ($[e]$). The relations of all 22 derived units listed in table 4 of the SI Brochure is shown in table 2.

For many of the SI derived units the relationships to the defining constants are simpler than to the SI base units.

It is worth noting that some of the relationships exhibit some of the fundamental quantum-physical effects. The SI unit for energy or work, the joule (J), relates to the defining constants by:

$$\mathrm{J} = [\Delta\upsilon_{\mathrm{Cs}}]\,[h]\,. \tag{1}$$

The well-known *Planck–Einstein relation* is directly visible in equation (1). The unit of the electric potential difference or voltage, the volt (V), relates to the defining constants by:

Table 1. SI base units with their proportional relationships to the defining constants, omitting the numerical scaling factors.

Base unit	Symbol	Relationship to the defining constants[a]
Second	s	$[\Delta\upsilon_{\mathrm{Cs}}]^{-1}$
Metre	m	$[\Delta\upsilon_{\mathrm{Cs}}]^{-1}\,[c]$
Kilogram	kg	$[\Delta\upsilon_{\mathrm{Cs}}]\,[c]^{-2}\,[h]$
Ampere	A	$[\Delta\upsilon_{\mathrm{Cs}}]\,[e]$
Kelvin	K	$[\Delta\upsilon_{\mathrm{Cs}}]\,[h]\,[k]^{-1}$
Mole	mol	$[N_{\mathrm{A}}]^{-1}$
Candela	cd	$[\Delta\upsilon_{\mathrm{Cs}}]^{2}\,[h]\,[K_{\mathrm{cd}}]$

[a] $[x]$ represents the unit of quantity x.

$$\mathrm{V} = [\Delta\upsilon_{\mathrm{Cs}}]\,[h]\,[e]^{-1}. \tag{2}$$

In this relationship the inverse of the *Josephson Constant* $K_{\mathrm{J}}^{-1} = h\,(2e)^{-1}$ is easily identified. Utilizing the Josephson effect high precision voltage standards can be created that directly realize the volt according to equation (2).

Finally, the unit of the electric resistance, the ohm (Ω), relates to the defining constant by:

$$\Omega = [h]\,[e]^{-2}. \tag{3}$$

In this relationship the *von Klitzing Constant* $R_{\mathrm{K}} = h\,e^{-2}$ is obvious, highlighting the possibility of realising the ohm directly through the quantum Hall effect.

There are other relationships where the outcome of the approach shown in table 2, whilst correct, has the result of disguising likely routes to direct realisation. The unit of absorbed dose, the gray (Gy), is an example of this. The analysis demonstrates that $\mathrm{Gy} = [c]^2$ but this does not clearly suggest a route for realisation of a quantity that is more usually conceived of as energy absorbed per mass.

The direct linking of the SI units with the defining constants can also be used in teaching and educational activities to illustrate the advantages of the revised SI. As an example, figure 1 illustrates the relationship of the ohm to the SI base units, whereas figure 2 illustrates the relationship of the ohm to the defining constants.

Also, in this representation, all SI units are equivalent—in that they can be directly realised using the defining constants—and hence, from the point of view of the laws of physics, specific connection to the SI base units is unnecessary. This last point can also be illustrated using the photometric units, as the defining constant, K_{cd}, links the photometric quantities and the associated units directly to the corresponding radiometric quantities. For monochromatic radiation of frequency 540×10^{12} Hz:

$$1\,\mathrm{lm} = \frac{K_{\mathrm{cd}}}{\{K_{\mathrm{cd}}\}}\,\mathrm{W}, \tag{4a}$$

$$1\,\mathrm{lx} = \frac{K_{\mathrm{cd}}}{\{K_{\mathrm{cd}}\}}\,\mathrm{W\,m^{-2}}, \tag{4b}$$

$$1\,\mathrm{cd} = \frac{K_{\mathrm{cd}}}{\{K_{\mathrm{cd}}\}}\,\mathrm{W\,sr^{-1}}, \tag{4c}$$

Table 2. The 22 SI derived units with special names and symbols according table 4 of the SI Brochure [3] with their relationships to the defining constants.

Derived quantity	Special name of unit	Unit expressed in terms of base units	Relationship to the defining constants[a]
Plane angle	Radian	$\mathrm{rad} = \mathrm{m\ m^{-1}}$	See footnote [b]
Solid angle	Steradian	$\mathrm{sr} = \mathrm{m^2\ m^{-2}}$	See footnote [b]
Frequency	Hertz	$\mathrm{Hz} = \mathrm{s^{-1}}$	$[\Delta\upsilon_{\mathrm{Cs}}]$
Force	Newton	$\mathrm{N} = \mathrm{kg\ m\ s^{-2}}$	$[\Delta\upsilon_{\mathrm{Cs}}]^2\,[c]^{-1}\,[h]$
Pressure, stress	Pascal	$\mathrm{Pa} = \mathrm{kg\ m^{-1}\ s^{-2}}$	$[\Delta\upsilon_{\mathrm{Cs}}]^4\,[c]^{-3}\,[h]$
Energy, work, amount of heat	Joule	$\mathrm{J} = \mathrm{kg\ m^2\ s^{-2}}$	$[\Delta\upsilon_{\mathrm{Cs}}]\,[h]$
Power, radiant flux	Watt	$\mathrm{W} = \mathrm{kg\ m^2\ s^{-3}}$	$[\Delta\upsilon_{\mathrm{Cs}}]^2\,[h]$
Electric charge	Coulomb	$\mathrm{C} = \mathrm{A\ s}$	$[e]$
Electric potential difference	Volt	$\mathrm{V} = \mathrm{kg\ m^2\ s^{-3}\ A^{-1}}$	$[\Delta\upsilon_{\mathrm{Cs}}]\,[h]\,[e]^{-1}$
Capacitance	Farad	$\mathrm{F} = \mathrm{kg^{-1}\ m^{-2}\ s^4\ A^2}$	$[\Delta\upsilon_{\mathrm{Cs}}]^{-1}\,[h]^{-1}\,[e]^2$
Electric resistance	Ohm	$\Omega = \mathrm{kg\ m^2\ s^{-3}\ A^{-2}}$	$[h]\,[e]^{-2}$
Electric conductance	Siemens	$\mathrm{S} = \mathrm{kg^{-1}\ m^{-2}\ s^3\ A^2}$	$[h]^{-1}\,[e]^2$
Magnetic flux	Weber	$\mathrm{Wb} = \mathrm{kg\ m^2\ s^{-2}\ A^{-1}}$	$[h]\,[e]^{-1}$
Magnetic flux density	Tesla	$\mathrm{T} = \mathrm{kg\ s^{-2}\ A^{-1}}$	$[\Delta\upsilon_{\mathrm{Cs}}]^2[c]^{-2}\,[h]\,[e]^{-1}$
Inductance	Henry	$\mathrm{H} = \mathrm{kg\ m^2\ s^{-2}\ A^{-2}}$	$[\Delta\upsilon_{\mathrm{Cs}}]^{-1}\,[h]\,[e]^{-2}$
Celsius temperature	Degree Celsius	$^\circ\mathrm{C} = \mathrm{K}$	$[\Delta\upsilon_{\mathrm{Cs}}]\,[h]\,[k]^{-1}$
Luminous flux	Lumen	$\mathrm{lm} = \mathrm{cd\ sr}$	$[\Delta\upsilon_{\mathrm{Cs}}]^2\,[h]\,[K_{\mathrm{cd}}]$
Illuminance	Lux	$\mathrm{lx} = \mathrm{cd\ sr\ m^{-2}}$	$[\Delta\upsilon_{\mathrm{Cs}}]^4[c]^{-2}\,[h]\,[K_{\mathrm{cd}}]$
Activity referred to a radionuclide	Becquerel	$\mathrm{Bq} = \mathrm{s^{-1}}$	$[\Delta\upsilon_{\mathrm{Cs}}]$
Absorbed dose, kerma	Gray	$\mathrm{Gy} = \mathrm{m^2\ s^{-2}}$	$[c]^2$
Dose equivalent	Sievert	$\mathrm{Sv} = \mathrm{m^2\ s^{-2}}$	$[c]^2$
Catalytic activity	Katal	$\mathrm{kat} = \mathrm{mol\ s^{-1}}$	$[\Delta\upsilon_{\mathrm{Cs}}]\,[N_{\mathrm{A}}]^{-1}$

[a] $[x]$ represents the unit of quantity x.
[b] For reasons of history and convention, plane and solid angles are treated within the SI as quantities with the unit one and so are independent of the choice of defining constants.

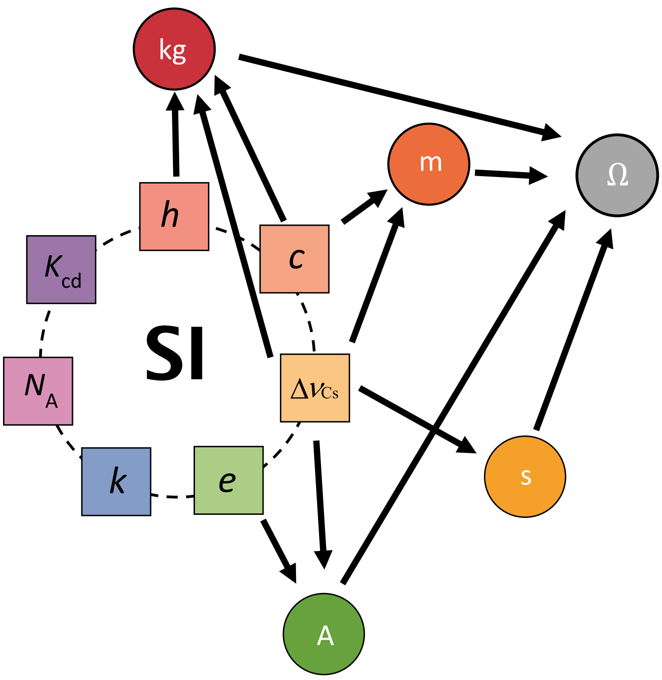

Figure 1. Schematic representation of the relationship between the ohm and the SI base units, $\Omega = \mathrm{kg\,m^2\,s^{-3}\,A^{-2}}$.

where $\{K_{\mathrm{cd}}\}$ indicates the numerical value of K_{cd}. In these fundamental relationships the base unit candela has no special role.

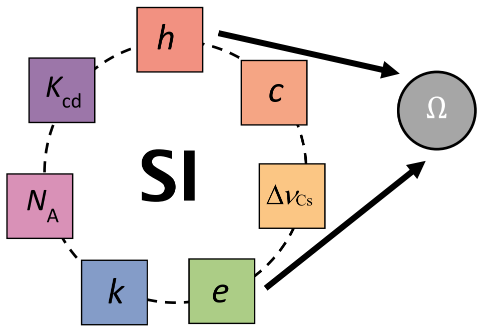

Figure 2. Schematic representation of the relationship between the ohm and the defining constants, $\Omega = [h]\,[e]^{-2}$.

3. Quantity calculus and dimensional analysis

Expressing derived units or combination of units in terms of base units provides a useful tool for checking the consistency of equations involving physical quantities, although it is important to note that it may not unambiguously distinguish between different kind of quantities that share the same dimensions, such as torque and energy. The validity of the relationships between quantities and units can also be confirmed by directly associating the SI units with the defining constants. As an example, the electrical potential energy, $U_{\mathrm{E}}(\boldsymbol{r})$, of a one point charge q at position $\boldsymbol{r}$ in the presence of an electric potential $V(\boldsymbol{r})$ is defined as the product of the charge and the electric potential:

$$U_{\mathrm{E}}(\boldsymbol{r}) = q \cdot V(\boldsymbol{r}). \tag{5}$$

To verify dimensional correctness the units of the left and right side are usually compared:

$$[U_{\mathrm{E}}] \stackrel{?}{=} [q] \cdot [V] \tag{6a}$$

$$\mathrm{J} \stackrel{?}{=} \mathrm{C} \cdot \mathrm{V}. \tag{6b}$$

A way to verify the correctness of the unit equation is to express both sides in terms of SI base units:

$$\mathrm{kg\,m^2\,s^{-2}} \stackrel{?}{=} (\mathrm{A\ s}) \cdot \left(\mathrm{kg\,m^2 s^{-3}\,A^{-1}}\right). \tag{7}$$

This equation is obviously correct. An alternative way to verify the correctness of equation (6) is to express the units in terms of their relationship to the defining constants:

$$[\Delta\upsilon_{\mathrm{Cs}}]\ [h] \stackrel{?}{=} [e] \cdot \left([\Delta\upsilon_{\mathrm{Cs}}]\ [h]\ [e]^{-1}\right). \tag{8}$$

The latter equation is also obviously correct, illustrating the possibility of verifying unit consistency by relating the units to the defining constants, without the specific use of SI base units.

4. Conclusion

Historically metrology was focused on the improvement of the primary realization of SI base units, especially when these related to unique physical artefacts. The concept of SI base units is still extensively used today to illustrate the fields of activity in metrology and for pedagogy. Today, metrology research typically done at National Metrology Institutes (NMIs) is much broader: it is concerned with the realisation of all SI units. For the historical reasons described above, the derived units have sometimes, wrongly, been seen as of secondary importance within and outside NMIs. Demonstrating the relationships between all the SI units and the defining constants illustrates that, in respect of their ability to be directly realised from the defining constants, there is little practical difference between the SI base units and SI derived units. This perspective may contribute to National Metrology Institutes being able to craft a more compelling narrative in future about the breath and impact of their work. This approach also allows quantum-physical principles to be neatly illustrated, and the consistency of the quantity and unit relationships to be easily verified.

Acknowledgments

R J C B gratefully acknowledges the funding of the National Measurement System by the UK Department for Science, Innovation and Technology.

ORCID iDs

Peter Blattner https://orcid.org/0000-0003-0681-2096
Richard J C Brown https://orcid.org/0000-0001-6106-0996

References

[1] BIPM 1970 *The International System of Units ('The SI Brochure')* (Bureau international des poids et mesures)
[2] Stock M, Davis R, de Mirandés E and Milton M J T 2019 The revision of the SI—the result of three decades of progress in metrology *Metrologia* **56** 022001
[3] BIPM 2019 *The International System of Units ('The SI Brochure')* 9th edn (Bureau international des poids et mesures)
[4] Brown R J C 2023 On the distinction between SI base units and SI derived units *Metrologia* **61** 013001
[5] Mohr P J 2008 Defining units in the quantum based SI *Metrologia* **45** 129–33
[6] Davis R S 2018 How to define the units of the revised SI starting from seven constants with fixed numerical values *J. Res. Natl Inst. Stand. Technol.* **123** 1–11