

Formulae for the calculation of the magnetic field strength in a Helmholtz system with circular coils

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ABSTRACT

Formulae are given for the magnetic field strength H of a Helmholtz system of two circular coils. The components of H along and perpendicular to the axis are expressed in series of spherical harmonics which converge rapidly in a region around the centre of the system.

The formulae are normalised, and thus applicable to systems of any size.

The results given in this Report are well known; they have been put together for easy reference.

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1 INTRODUCTION

Fig 1 shows two identical coils of radius a carrying a current I . It is shown below that the greatest uniformity of field near the centre of the system is provided when the separation $2b$ of the coils is made equal to their radius a . For the moment b will be left unspecified.

2 NORMALISATION

It is convenient to express all distances in terms of the radius a , writing b, c, y, z , etc $\beta a, \gamma a, \eta a, \zeta a$, etc, where $\beta, \gamma, \eta, \zeta$, etc are mere numbers. This process of "normalisation" renders applicable to all sizes formulae describing the field of a system of a particular shape.

3 MAGNETIC SCALAR POTENTIAL

The magnetic scalar potential, here denoted by V , is a function whose gradient ∇V taken with a negative sign, yields the magnetic field strength H . It is suitable for application to configurations with axial symmetry like the above.

In fig 1, if Ω is the solid angle that the circle at A subtends at Q on the axis, the potential is given by

$$V = \frac{\Omega I}{4\pi} \quad (1)$$

According to the conventions for direction of current and for magnetic field strength, I is considered positive if it is seen to be flowing counterclockwise when the circle at A is viewed from the point Q, fig 1; if it is seen to flow clockwise, as for the circle at B viewed from Q, it must be given a negative sign in (1).

The solid angle Ω is that part of the area of the surface of the sphere of radius QC bounded by the circle, divided by QC^2 . It is given by

$$\Omega = 2\pi (1 - \cos AQC) \quad (2)$$

Denote by V_1 and V_2 the potentials at Q of the coils at A and B. From (1) and (2)

$$V = \frac{I}{2} (1 - \cos AQC)$$

$$\frac{I}{2} \left[1 - \frac{b+y}{\sqrt{a^2 + (b+y)^2}} \right]$$

$$\frac{I}{2} [1 - (\beta+\eta)(1+\beta^2+2\beta\eta+\eta^2)^{-1/2}] \quad (3)$$

$$V_2 = -\frac{I}{2}(1 - \cos BQD)$$

$$-\frac{I}{2} \left[1 - (\beta - \eta)(1 + \beta^2 - 2\beta\eta + \eta^2)^{-1/2} \right] \quad (4)$$

The potential V , given by the sum of (3) and (4) is then

$$V = V_1 + V_2$$

4 SPHERICAL HARMONICS

Any solution of Laplace's equation $\nabla^2 V=0$ is said to be a "spherical harmonic" (Jeans 1927 p.208). Jeans (1927 pp.206-242 and 431-432) gives a good treatment of those functions; see also Gray (1921). They are particularly well suited to problems with axial symmetry, because once the potential at a point Q on the axis has been established as a series of spherical harmonics in powers of OQ , or y , the potential at a distance r , θ from the origin is obtained by replacing y^n by $r^n P_n(\cos \theta)$. Appendix 1 gives formulae for spherical harmonics and their derivatives; see also Dwight (1949 pp.192-193) who calls them "surface zonal harmonics", and other books of mathematical tables.

The functions P_n , see Appendix 1, are "Legendre polynomials", usually expressed in terms of $\cos \theta$. Another set of these coefficients will be required in this note; they will be denoted by $Z_n(\cos \alpha)$ to provide easy distinction when the arguments $\cos \theta$ and $\cos \alpha$ are omitted for brevity. Z' and P' are derivatives with respect to $\cos \alpha$ and $\cos \theta$, not with respect to the angles. It will sometimes be convenient to write v for $\cos \alpha$, and μ for $\cos \theta$.

5 SERIES EXPANSION OF V

Formula (4) can be written (see fig 1)

$$V_2 = -\frac{I}{2} \left[1 - \left(\cos \alpha - \frac{\eta}{\gamma} \left(1 - 2\frac{\eta}{\gamma} \cos \alpha + \frac{\eta^2}{\gamma^2} \right)^{-1/2} \right) \right] \quad (5)$$

Jeans (1927, p.217) shows that if $\eta < \gamma$ the reciprocal square root can be expanded in a series of harmonics thus:

$$\left(1 - \frac{2\eta}{\gamma} \cos \alpha + \frac{\eta^2}{\gamma^2} \right)^{-1/2} = 1 + \frac{\eta}{\gamma} Z_1(v) + \dots + \frac{\eta^n}{\gamma^n} Z_n(v) + \dots \quad (6)$$

If $\eta > \gamma$ the corresponding series is

$$\left(1 - \frac{2\eta}{\gamma} \cos \alpha + \frac{\eta^2}{\gamma^2} \right)^{-1/2} = \frac{\gamma}{\eta} + \frac{\gamma^2}{\eta^2} Z_1(v) + \dots + \frac{\gamma^{n+1}}{\eta^{n+1}} Z_n(v) + \dots \quad (7)$$

(6) applies to points between the coils whereas (7) is applicable to points outside the system. If however the distance at which the potential is needed exceeds ten times the radius of the coils, it may be simpler to substitute for the coils their equivalent magnetic moment (see Jeans 1927 p.425).

In this Report we confine our interest to points between the coils, and use expansion (6). From (3) and (6)

$$V_1 = \frac{I}{2} \left[1 - v + \frac{\eta}{\gamma} (vZ_1 - 1) + \dots + (-)^{n-1} \frac{\eta^n}{\gamma^n} (vZ_n - Z_{n-1}) + \dots \right] \quad (8)$$

and from (4) and (6).

$$V_2 = \frac{I}{2} \left[-1 + v + \frac{\eta}{\gamma} (vZ_1 - 1) + \dots + \frac{\eta^n}{\gamma^n} (vZ_n - Z_{n-1}) + \dots \right] \quad (9)$$

$$V = V_1 + V_2 = I \left[\frac{\eta}{\gamma} (vZ_1 - 1) + \dots + \frac{\eta^{2n+1}}{\gamma^{2n+1}} (vZ_{2n+1} - Z_{2n}) + \dots \right]$$

The terms in brackets in (10) can be simplified by introducing the relation (see Appendices 2 and 3).

$$vZ_{2n+1} - Z_{2n} = -\frac{\sin^2 \alpha}{2n+1} Z'_{2n+1}$$

to obtain

$$V = -I \sin^2 \alpha \sum_0^{\infty} \frac{\eta^{2n+1}}{\gamma^{2n+1}} \frac{Z'_{2n+1}}{2n+1} \dots$$

6 OPTIMUM SEPARATION OF COILS

The magnetic field strength H at Q , fig 1, is $-dV/dy$. From (12) and fig 1

$$H = -\frac{1}{a} \frac{dV}{d\eta} = \frac{I}{a} \sin^2 \alpha \sum_0^{\infty} \frac{\eta^{2n}}{\gamma^{2n+1}} Z'_{2n+1} \dots \quad (13)$$

The field strength at the centre O is the first term of (13), ie

$$H = \frac{I \sin^2 \alpha}{a\gamma} \quad (14)$$

The second term (see Appendix 1) is

$$\delta H = \frac{3I\eta^2}{2a\gamma^3} \sin^2 \alpha (5 \cos^2 \alpha - 1) \quad (15)$$

This term is the main cause of variation of H along the axis. If α , or b , are chosen for it to vanish, uniformity is much improved. The value zero for α is irrelevant, for it would make the separation b infinite, and although there would be no variation near O , there would also be no field there. Thus $\cos^2 \alpha$ must be made $1/5$; in consequence b is $a/2$, γ is $\sqrt{5}/2$, and the separation is equal to the radius a .

The field strength at the centre is then

$$H_c = \frac{I}{a} \frac{8}{5\sqrt{5}}$$

The third term

$$\delta\delta H = \frac{I}{a} \frac{204}{5^{5.5}} \eta^4$$

becomes the main cause of lack of uniformity at points on the axis between coils with optimum separation.

If H_A is the magnetic field strength at A, fig 1, the relative departure from the strength H_c at the centre is then

$$\frac{H_A - H_c}{H_c} = \frac{51}{2} \times 10^{-4} \quad (18)$$

At a tenth of the distance of the coil from the centre, η is $1/20$, and the corresponding departure is only $\frac{51}{2} \times 10^{-8}$.

7 EXPANSION OF V OFF THE AXIS

As explained in Section 5, the magnetic scalar potential at P, fig 1, of coordinates r, θ , or $\rho a, \theta$, is obtained from (12) by replacing η^{2n+1} by $\rho^{2n+1} P_{2n+1}(\cos \theta)$. Using the values $2/\sqrt{5}$ for $\sin \alpha$ and $1/\gamma$ given in section 6, we find for the potential

$$V(r, \theta) = -I \frac{4}{5} \sum_0^{\infty} \left(\frac{2\rho}{\sqrt{5}} \right)^{2n+1} \frac{Z'_{2n+1}}{2n+1} P_{2n+1}(\cos \theta)$$

8 MAGNETIC FIELD STRENGTH OFF THE AXIS

The axial and radial components of H at r, θ are

$$H_y = -\frac{dV}{dy} = -\frac{1}{a} \frac{dV}{d\eta} \quad (20)$$

$$H_z = -\frac{dV}{dz} = -\frac{1}{a} \frac{dV}{d\zeta} \quad (21)$$

which, expressed in coordinates ρ, θ become

$$H_y = -\frac{1}{a} \left(\cos \theta \cdot \frac{dV}{d\rho} - \frac{\sin \theta}{\rho} \frac{dV}{d\theta} \right) \quad (22)$$

$$H_z = -\frac{1}{a} \left(\sin \theta \cdot \frac{dV}{d\rho} + \frac{\cos \theta}{\rho} \frac{dV}{d\theta} \right) \quad (23)$$

Because of circular symmetry, the Z direction QP, fig 1, may be taken parallel to any radius of either coil. The magnetic field strength H_z is the same at all points of the circumference of a circle of centre Q and radius QP; it vanishes at all points of the Y axis.

From (16), (22) and (19),

$$\begin{aligned}
 H_y &= H_o \left[\cos \theta \sum_0^{\infty} \left(\frac{2\rho}{\sqrt{5}} \right)^{2n} \cdot Z'_{2n+1} \cdot P_{2n+1} + \sin^2 \theta \sum_0^{\infty} \left(\frac{2\rho}{\sqrt{5}} \right)^{2n} \cdot Z'_{2n+1} \cdot \frac{1}{2n+1} \cdot P'_{2n+1} \right] \\
 &= H_o \sum_0^{\infty} \left(\frac{2\rho}{\sqrt{5}} \right)^{2n} \cdot Z'_{2n+1} \cdot P_{2n}
 \end{aligned} \tag{24}$$

(For the last step see Appendix 3, first formula).

From (16), (23) and (19),

$$\begin{aligned}
 H_z &= H_o \left[\sin \theta \sum_0^{\infty} \left(\frac{2\rho}{\sqrt{5}} \right)^{2n} \cdot Z'_{2n+1} \cdot P_{2n+1} + \cos \theta \sum_0^{\infty} \left(\frac{2\rho}{\sqrt{5}} \right)^{2n} \cdot Z'_{2n+1} \cdot \frac{1}{2n+1} \cdot -\sin \theta \cdot P'_{2n+1} \right] \\
 &\quad - H_o \sin \theta \cdot \sum_0^{\infty} \left(\frac{2\rho}{\sqrt{5}} \right)^{2n} \cdot Z'_{2n+1} \cdot \frac{1}{2n+1} \cdot P'_{2n}
 \end{aligned} \tag{25}$$

(See Appendix 3, second formula).

9 NORMALISED EXPRESSIONS FOR H

The variation of the strength H of the magnetic field in the space between the coils is best expressed by the values of its components H_y and H_z along and perpendicular to the axis, relative to the value of H at the centre O, fig 1. There, by symmetry, H_z vanishes, as it does at all points of the axis, and H_y is given by (16).

From (24) and (25), the relative changes R_y and R_z at the point r, θ are

$$\begin{aligned}
 R_y &= \frac{H_y - H_o}{H_o} = \sum_0^{\infty} \left(\frac{2\rho}{\sqrt{5}} \right)^{2n} \cdot Z'_{2n+1} \cdot P_{2n} - 1 \\
 R_z &= \frac{H_z}{H_o} = -\sin \theta \sum_0^{\infty} \left(\frac{2\rho}{\sqrt{5}} \right)^{2n} \cdot Z'_{2n+1} \cdot P'_{2n} \cdot \frac{1}{2n+1}
 \end{aligned}$$

These series converge the more rapidly the closer P is to O, fig 1

10 ALLOWANCE FOR CROSS-SECTION OF WINDINGS

The formulae developed hitherto involve the assumption that the "coils" at A and B, fig 1, are single circles of wire of negligible cross-section. In practice they consist of a large number of turns of wire of adequate cross-section to carry the required current without undue heating, wound in grooves of width, say w , and depth d . Gray (1921 p.220) shows that instead of (15), the condition for δH to vanish is, leaving out a common factor,

$$\frac{5b^2}{c^2} - 1 + \frac{4}{75a^2} (31d^2 - 36w^2) = 0 \quad (28)$$

See also NPL notebook PV 1958 GM3, p.19.

One way of securing this condition for optimum separation is to make the turns per layer equal to the number of layers, and separate the layers by plastic ribbon of thickness such as to increase the depth by the required amount. Although this solution, making the two terms of (28) vanish independently, is perhaps the more elegant, it risks being impracticable if the material of the coil formers is available only in standard thicknesses. The alternative is to reduce the second term as far as is possible with standard material, and change the coil separation $2b$ by the amount necessary to satisfy (28).

As an example, the dimensions of the H compensating coils of the gyromagnetic ratio apparatus in Room 21, Bushy House, are, in mm:

$$a = 1120 \quad w = 9.6 \quad d = 3.6$$

The normal separation a of the coils was increased by 0.087 mm in order to satisfy (28)

The corresponding increase for the V coils is 0.063 mm

See NPL notebook PV 1958 GM3, pp.2-23.

11 REFERENCES

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APPENDIX 1 LEGENDRE FUNCTION

$$P_n(\cos \theta) = P_n(\mu) = \frac{1}{2^n n!} \left(\frac{\partial}{\partial \mu} \right)^n (\mu^2 - 1)^n$$

$$P_0 \quad 1$$

$$P_1 \quad \mu$$

$$P_2 \quad \frac{1}{2} (3\mu^2 - 1)$$

$$P_3 \quad \frac{1}{2} (5\mu^3 - 3\mu)$$

$$P_4 \quad \frac{1}{8} (35\mu^4 - 30\mu^2 + 3)$$

$$P_5 \quad \frac{1}{8} (63\mu^5 - 70\mu^3 + 15\mu)$$

$$P_6 \quad \frac{1}{16} (231\mu^6 - 315\mu^4 + 105\mu^2 - 5)$$

$$P_7 \quad \frac{1}{16} (429\mu^7 - 693\mu^5 + 315\mu^3 - 35\mu)$$

$$P'_n(\cos \theta) = P'_n(\mu) = \frac{d}{d\mu} P_n(\mu)$$

$$P'_0 \quad 0$$

$$P'_1 \quad$$

$$P'_2 \quad 3\mu$$

$$P'_3 \quad \frac{1}{2} (15\mu^2 - 3)$$

$$P'_4 \quad \frac{1}{8} (140\mu^3 - 60\mu)$$

$$P'_5 \quad \frac{1}{8} (315\mu^4 - 210\mu^2 + 15)$$

$$P'_6 \quad \frac{1}{8} (693\mu^5 - 630\mu^3 + 105\mu)$$

$$P'_7 \quad \frac{1}{16} (3003\mu^6 - 3465\mu^4 + 945\mu^2 - 35)$$

APPENDIX 2 RECURRING FORMULAE FOR LEGENDRE FUNCTION

$$(n+1) P_{n+1} = (2n+1) \mu P_n - n P_{n-1}$$

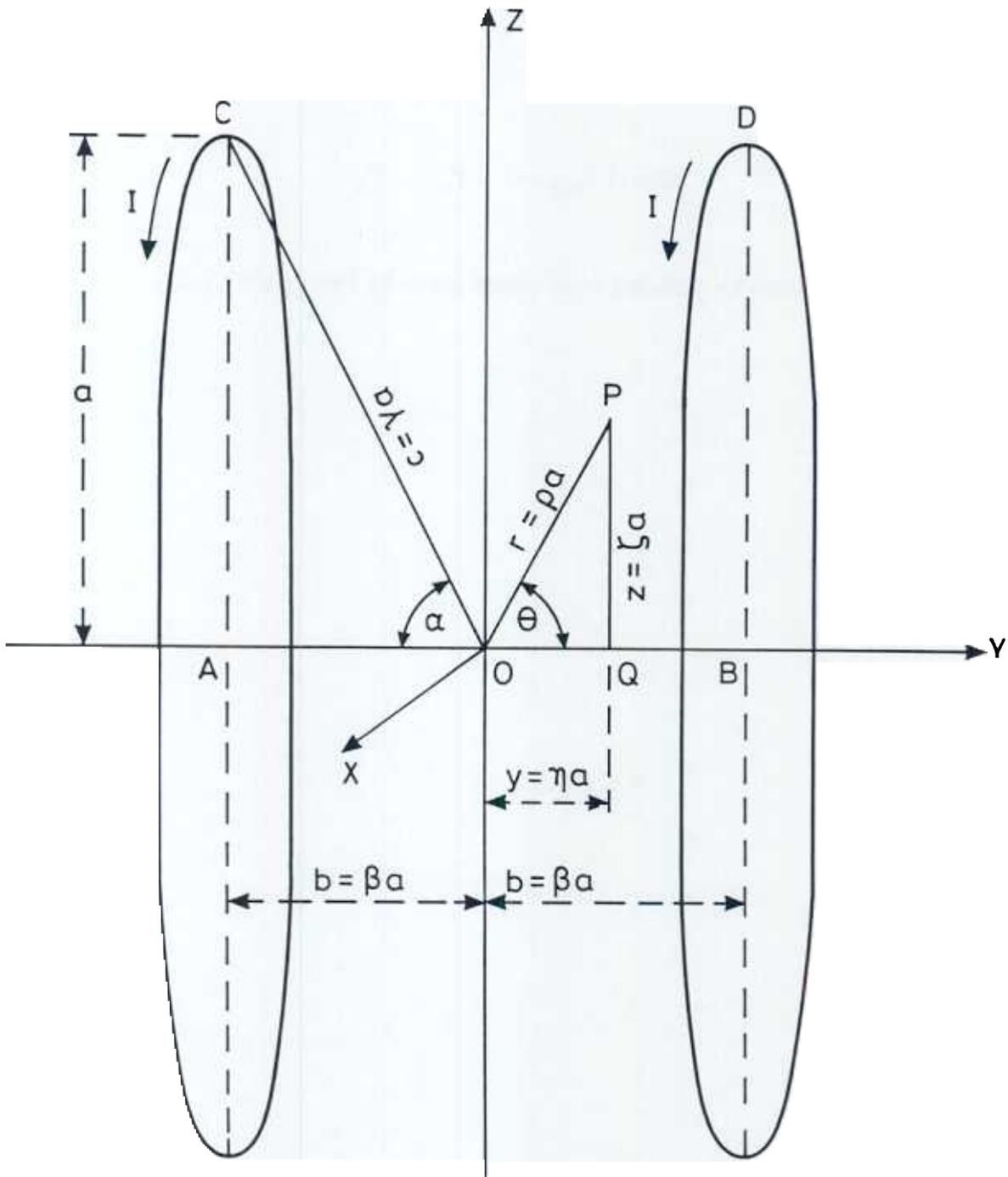
$$n P'_{n+1} = (2n+1) \mu P'_n - (n+1) P'_{n-1}$$

APPENDIX 3 OTHER FORMULAE USED IN THIS REPORT

$$\mu P_{2n+1} - P_{2n} = \frac{\mu^2 - 1}{2n+1} P'_{2n+1}$$

$$(2n+1) P_{2n+1} = \mu P'_{2n+1} - P'_{2n}$$

These two equalities can be deduced from others given by Jeans (1927), p.221.



$$\frac{b}{c} = \frac{\beta}{\gamma} = \cos \alpha = v$$

FIGURE 1: NOTATION FOR DIMENSIONS OF HELMHOLTZ SYSTEM