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Statistical characterization of tailored complex composite media for certified reference materials

L R Arnaut

Center for Electromagnetic and Time Metrology

National Physical Laboratory

Teddington, Middlesex TW11 0LW, United Kingdom

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Abstract

This report addresses the statistical macroscopic characterization of composite materials in simple and complex electromagnetic environments. Based on the application of the central limit theorem to the random parameters that characterize the microscopic constituents and their configuration, the distribution function, mean and uncertainty for various macroscopic effective constitutive material and wave parameters are derived. The implications of the randomness of the external field on these parameters are also investigated.

Keywords: Complex composite media, certified reference materials, statistical characterization, measurement uncertainty, sample tolerances, effective medium, homogenization, strong-property fluctuation, Dyson equation, random fields, signal diversity, fading.

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National Physical Laboratory
Teddington, Middlesex TW11 0LW, United Kingdom

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Stuart Pollitt, Head of the Center for Electromagnetic and Time Metrology

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1 Introduction

1.1 Statistics of specimens of composite materials

Heterogeneous media containing discrete particles of various types and arrangements involve many intricacies in their rigorous modelling and characterization, theoretically as well as experimentally. Their eligibility as benchmark or certified reference materials (CRMs) justifies their investigation for the purpose of material and instrumentation calibration, beyond the existing wide interest in them for other purposes and applications. Such composite specimens can be devised to yield a set of tailored and calculable material systems which exhibit parameters with specified nominal mean values and associated traceable uncertainty and confidence levels.

In analytical electromagnetic modelling, composite media are usually idealized as being infinite and unbounded. The particles can then be considered to be of zero scale, relative to the scale of the medium, and countably infinite in number. As a result, the volume density of particles is finite and each particle can be modelled, in the quasi-static approximation, as a point-polarizable entity. Under these assumptions the homogenization procedure in effective medium theory is exact, asymptotically (i.e. in the long-wavelength limit), and applies to random arrangements as well as regular lattices of particles. As a direct consequence of its infinite extent, any local statistical deviation from nominal randomness or regularity of any parameter of the composite medium will occur with zero weight in the relevant effective property of the infinite medium, even for finite volume densities, because the total number of particles is infinite. The constitutive description of the infinite medium is subsequently conveyed to that of finite-sized material specimens (samples), by a posteriori imposing appropriate electromagnetic continuity conditions at its boundaries. However, the homogenization procedure when applied to a composite specimen of finite extent is inevitably approximate, because the finiteness of the macrosopic scale of the sample renders the assumption of zero-scale particles relative to the scale of

the medium now as being approximate. In this case a characterization based on statistical averages only (first-order moments) is deficient. [The issue is supplementary to the effects of finite size of particles themselves relative to the ambient wavelength (spatial disperison).] This question regarding the accuracy and applicability of the homogenization procedure in practice is particularly relevant to miniaturized electronic circuit components (MMICs) and thin molecular layers or substrates. For the latter case, the study of surface effects on effective medium characteristics has a long history and has recently triggered renewed interest, e.g. [1, 2].

Two issues closely related to finite sample size are intra- and inter-sample variability with regard to the composition, shape, size, orientation, configuration etc. of the particles. Especially the inter-sample variations, whether occurring between specimens originating from different batches or from a single batch, are of major concern in estimating the uncertainty budget and the representativity of small samples in the characterization of the bulk composite (yield). Here the term 'variability' is used in the wide sense, inclusive of the uncertainty level with which nominally identical particles can be characterized. Furthermore, the degree of randomness of the medium assumed in this paper is more general than what is often being denoted as a random medium, because we consider *any* degree of deviation from a perfectly regular, symmetric (i.e. crystallographic) lattice of particles.

In principle, a rigorous treatment based on analytical (T-matrix, de-embedding [3, 4], etc.) or full-wave numerical [5] procedures for a specific sample yields the correct answer, but this is usually of almost forbidding complexity, or even impossible if the physical, geometrical or configurational properties of the particles are not known with certainty. The effects of ignoring the uncertainty or variability of some microscopic characteristic may propagate and even escalate into the macroscopic description. However, the *exact* details of the intra- and inter-sample variability should not be relevant to the macroscopic characterization of the sample in question, for effective medium modelling to be practically useful at all. In

view of this fact, statistical modelling of the sample offers an efficient and simple alternative to 'exact' methods, along with additional information on uncertainties for effective medium theories. An attractive feature of statistical modelling is that the uncertainty on the mean effective medium parameters *decreases* inversely proportional with the square root of the number of variants (particles), i.e. with the complexity of the medium. In electromagnetics of macroscopically homogeneous media, containing a very large number of particles ('atoms') at the microscopic scale, such as in crystals and nano-arrays at relatively long wavelengths, an extreme kurtosis of the sample probability density function (pdf) exists. As this number further increases, the pdf then tends (non-analytically) towards a Dirac delta distribution. This then justifies the restriction on the statistical characterization to a single statistic, viz. the sample mean. On the other hand, reducing the particle density gives rise to larger uncertainties, hence the specimen is then progressively less efficiently modelled using a single statistic, requiring the specification of additional moments of the pdf. As the particle concentration decreases to very low levels, the statistical description becomes insufficient altogether. Thus, the full statistical characterization discussed in this paper is useful for medium particle loading fractions $\mathcal{N}$ within a sample volume $\mathcal{V}$, i.e. when the number of particles is too large for 'exact' deterministic methods to be practical, but not so large that a merely marginal increase in accuracy or definition, compared to the use of statistical mean values only, hardly justifies the increased mathematical overhead. As a guide, a full statistical analysis is recommended between $\mathcal{N}\mathcal{V} \sim 10^1$ and $\mathcal{N}\mathcal{V} \sim 10^3$, thus spanning a range of great practical interest. However, statistical uncertainties are significant not only in increasing the numerical accuracy in constitutive characterization, but also in establishing the qualitative significance of certain theoretically predicted or experimentally observed effects (e.g. natural nonreciprocity, nonlinearity, etc.). In such cases, tests of hypothesis and use of confidence intervals for the parameters of interest assist in the decision process.

1.2 Complex fields

In addition to the randomness of the medium itself, its excitation too may contain a source of variability or uncertainty, whether accidentally or intentionally. A growing trend exists towards the use of complex electromagnetic environments as alternative measurement sites, in order to characterize media and equipment for dedicated operational characteristics of propagation (e.g. indoor propagation, (non)dispersive multi-path fading, multi-source signal diversity, etc.). This yields more realistic or accurate results than those achieved using single plane-wave measurement fixtures. In the area of electromagnetic compatibility (EMC), one such field generator is known as a reverberation chamber, furnished with additional mechanical or electronic mode-tuning or mode-stirring capabilities [6, 7]. Here, excitations involve complex-shaped wave fronts, multiple reflections, and random direction of incidence and degree of wave polarization, either deterministic and predictable, or quasi- or fully random (electromagnetically chaotic). In the latter case, the incident wave may be considered as statistical, viz. originating from a spherically symmetric angular spectrum of random plane waves. Deterministic formulations in terms of eigenwave expansions are also in use, e.g. for focused beams or whirls [8]. A similar situation occurs for the field inside strongly scattering media, even if the externally incident wave is simple or deterministic. In the latter case, however, the randomness of the medium and of the internal wave are no longer statistically independent. For cavities with finite dimensions, the cavity field is necessarily only partially reverberant, i.e. a combination of partially coherent and partially incoherent waves. These hybrid fields are referred to as complex. The most general case is that of complex random media in complex fields.

2 Basic assumptions and definitions

We consider complex materials as defined implicitly in [9, Section 5.4.G], viz. electromagnetic media for which the Taylor series expansion of its constitutive equations is nonlinear

in the coordinates:

$$G^{(M)}(\underline{r}') = \sum_{i=1}^{M} (\underline{r}'_{K_1} - \underline{r}_{K_1}) \dots (\underline{r}'_{K_i} - \underline{r}_{K_i}) G_{K_1,\dots,K_i}(\underline{r})/i! \tag{1}$$

with $M > 1$, i.e. the nonlocality of a complex medium shows intra-particle as well as inter-particle functionality (short-range and long-range spatial dispersion). In (1), $G = \{\underline{r}_K, \underline{E}_K, \underline{B}_K\}$ is the minimum set of independent spatial and electromagnetic variables which fully characterize the medium and fields at local coordinates r_i for each charge i that makes up particle K, with $r_K^{(\prime)}$ denoting the location of the chosen center of the particle in the inertial frame $R_G^{(\prime)}$, with R_G and R'_G linked by a Galilean transformation. The constitutive relations are uniformly continuous response functionals of G. Simple materials, i.e. (1) with $M = 1$, exhibit first-order functional gradients in one or more dimensions [9]. The assumed non-relativistic approximation implies that the underlying multipole moments of all orders are assumed to be constant w.r.t. particle motion, unlike in a general covariant theory.

We assume each actual sample of the medium to be a particular realization of the ensemble (population) of samples which it represents. This realization process can be considered to take place in discrete time, hence each realized sample can be assigned a value of a discrete realization parameter τ. All parameters for which interaction is not explicitly taken into account in the model, viz. the particle density, permittivity of the host medium and individual particle dipolarizabilities, are taken to be independent so that their joint pdf equals the product of their respective marginal pdfs. This enables the effect of their respective uncertainties to be analyzed individually. The ensemble is considered to be thermodynamically stable, implying that each particular realization of the composite medium is equally likely. We further assume that the properties of this ensemble (discrete-time statistics) are the same as those of the unbounded infinite composite medium (spatial statistics), with possible exception of a set of weight zero, i.e. the composite medium is assumed to be ergodic in all its statistics in the strict sense [10, 11]. Therefore, the individual finite-sized samples may be considered to be 'cuts' from the infinite medium, or 'snapshots' in time

of a sample of a turbulent medium with moving particles. Wide-sense ergodicity, i.e. ergodicity for first- and second-order moments only, will usually suffice. This implies that the composite medium can be considered as stationary, in the strict or wide sense respectively, with respect to space and time so that its statistical properties are invariant with respect to the choice of the space-time reference frame. For example, the characteristics of a turbulent composite medium in a chosen spatial coordinate frame of reference are then invariant with respect to the time origin. In a solid-state medium, stationarity signifies that the value of any statistic of the medium is independent of the spatial origin against which it is calculated. The properties of stationarity and ergodicity make it possible to derive spectral statistics of a set of realized samples of the medium from those for the unbounded infinite medium and vice versa.

A three-dimensional spatial Fourier transformation of the composite medium yields information on possible local nonuniformities and its randomness in general. This characterization in the spatial spectral domain is usually more efficient than in real Euclidian space, because the details at high spatial frequencies are not relevant in effective medium theory, so that a band-limited description can be used.

Unlike e.g. [1, 2], where very large aspect ratios between the characteristic dimensions of the medium in the three dimensions are considered, resulting in intricacies with respect to the calculation of the local field, we shall here consider finite-sized samples with aspect ratios that are of the order of unity. The approach taken in this paper also differs from the method used in statistical physics, where kinetic-statistical methods for microscopic electromagnetism are used based on the Lorentz theory of electrons [9, 12, 13]. The latter approach requires knowledge of the joint pdfs of position, velocity and time of all elementary charge carriers involved in phase space, effectively giving a physical basis for, and bypassing the need for microscopic and macroscopic constitutive relations in describing material effects. Such distributions are seldom known in sufficient detail and with sufficient

accuracy, and the mathematical complexity and overhead becomes quickly insurmountable when explicit complete or practically relevant results are sought. Therefore, such approach is almost invariably being restricted to the calculation of mean values only. By contrast, the following approach relies on an intrinsic multi-particle description, calculable statistics of macroscopic constitutive parameters, the applicability of the central limit theorem and propagation of distributions. The results are cast in the framework of 'dielectric engineering' or 'molecular electronic engineering' [14]. The macroscopic characteristics of the medium and waves are obtained as pdfs, allowing for a full statistical characterization, in particular leading to quantification of uncertainties and confidence intervals.

As a general remark, the characteristics of second-order descriptors such as correlation and spectral density functions, yielding information on the spatial or temporal *rate* of fluctuations, are the focus in e.g. strong-property fluctuation theory. In this paper, we shall instead be concerned with the first-order probability distributions, which govern the statistics of the *magnitude* of the fluctuations. Also, we shall occasionally use standardized random variables $X' = (X - \mu_X)/\sigma_X$, to simplify the integrations. This poses no limitation on the practicality of the results, since the transformation from the pdf of X' to the pdf of X can always be obained as $f_X(x) = f_{X'}[(x - \mu_X)/\sigma_X]/\sigma_X$. We follow the notational convention of representing random variables as upper case symbols and the associated values for a particular realization as lower case symbols.

3 Complex media

3.1 Lorenz–Lorentz formula for random particles

The Maxwell–Garnett model for mixtures, when expressed in terms of the microscopic dipolarizabilities of the particles, gives rise to its Lorenz–Lorentz formulation. Here we extend this formula for the case where the geometrical and physical properties of the particles show statistical variations.

For a mixture of $\mathcal{NV}$ anisotropic dielectric particles in a volume $\mathcal{V}$ for an isotropic dielectric background medium ϵ_o, the electric dipole moment density is given by:

$$\underline{P} = \sum_{i=1}^{\mathcal{NV}} \underline{p}_i/\mathcal{V} = \sum_{i=1}^{\mathcal{N}} \underline{\underline{\alpha}}_i \cdot \underline{E}_{L_i}. \tag{2}$$

Since this concerns a particular realization, the random variables $\underline{P}$, $\mathcal{N}$, $\mathcal{V}$, $\underline{\underline{\alpha}}_i$ and $\underline{E}_{L_i}$ all depend on τ. The Mossotti local field $\underline{E}_{L_i}$ consists of the external incident field $\underline{E}$ (assumed to be constant for all particles), the near field $\underline{E}_{near_i}$, and the net internal field $\underline{\underline{L}}_i \cdot \sum_{j\neq i}^{\mathcal{NV}} \underline{p}_j/\epsilon_o$. We assume that $\underline{E}_{near_i}$ — which is zero for simple cubic and purely random lattice configurations — is not affected to first order by the randomization of the characteristics of the particle. This assumption is reasonable because the near field contains a linear *sum* of fields induced by neighbouring particles. For the same reason, we can approximate $\sum_{j\neq i}^{\mathcal{NV}} \underline{p}_j$ by $\sum_{i=1}^{\mathcal{NV}} \underline{p}_i$. With these assumptions, we obtain per realization τ:

$$\sum_{i=1}^{\mathcal{NV}} \underline{p}_i/\mathcal{V} \;=\; \left[\underline{\underline{I}} - \epsilon_o^{-1} \sum_{i=1}^{\mathcal{N}} \underline{\underline{L}}_i \cdot \left(\underline{\underline{I}} + \underline{\underline{\phi}}_i\right) \cdot \underline{\underline{\alpha}}_i\right]^{-1} \cdot \left[\sum_{i=1}^{\mathcal{N}} \underline{\underline{\alpha}}_i\right] \cdot \underline{E}, \tag{3}$$

where $\underline{\underline{\phi}}_i(\tau)$ accounts for the material structure and particle interaction (higher-order multipoles) at the microscopic level [15, 16]. This term is a result of the assumed complexity of the medium. Its functional form can be obtained, for example, using cluster expansion [17]–[21] or particle de-embedding [3, 4]. Hence,

$$\underline{\underline{\epsilon}} \;=\; \epsilon_o \underline{\underline{I}} + \left[\underline{\underline{I}} - \epsilon_o^{-1} \sum_{i=1}^{\mathcal{N}} \underline{\underline{L}}_i \cdot \left(\underline{\underline{I}} + \underline{\underline{\phi}}_i\right) \cdot \underline{\underline{\alpha}}_i\right]^{-1} \cdot \left[\sum_{i=1}^{\mathcal{N}} \underline{\underline{\alpha}}_i\right]. \tag{4}$$

With $\sum_i \underline{\underline{\alpha}}_i = \mathcal{N}\langle\underline{\underline{\alpha}}\rangle$, $\underline{\underline{\phi}}_i = \underline{\underline{0}}$ and $\underline{\underline{L}}_i = \underline{\underline{I}}/3$ this is *formally* identical to the classical Lorenz–Lorentz formula for mixtures involving spherical deterministic particles, but note that $\underline{\underline{\alpha}}_i$ and $\mathcal{N}$ are now random dyadic and random scalar variables, respectively, which constitutes a non-trivial extension. Thus, the classical Lorenz–Lorentz formula also applies, to first order, to the mean dipolarizability $\langle\underline{\underline{\alpha}}\rangle$ and mean particle density $\langle\mathcal{N}\rangle$. Provided that for each element α_{kl} of this sum dyadic the central limit theorem can be applied [22], the standardized sum corresponding to this element will be distributed according to

a Gauss normal distribution. This theorem can be extended to apply to *non-independent* random variables, provided that the correlation between particles and their characteristics is weak and that no single particle has a dominant contribution to the standard deviation of the overall sum [23]. This extension is of great practical interest in the application to complex materials, because inhomogeneities of the sample properties (in particular, bias with respect to the mean) often occurs in the form of aggregates (clusters) of particles. This is usually a result of the effect of gravity, variations in temperature, pressure etc. during manufacturing of the particles or during their insertion into the matrix, so that inhomogeneous correlations between the particles occur. The spatial correlation between particles can always be increased or decreased by 'clustering' or 'shuffling' the position of the particles in space, respectively. Hence, owing to the extended central limit theorem, $\sum_i \underline{\underline{\alpha}}_i$ can be statistically characterized in such systems with short-range interaction as well.

For the remainder we concentrate on the isotropic case, viz. a mixture of interacting particles embedded inside a host medium of permittivity $\epsilon_o(\tau)$ with $\underline{\underline{\alpha}}_i(\tau) \stackrel{\Delta}{=} \alpha_i(\tau)\underline{\underline{I}}$. Thus, for nominally identical particles:

$$\epsilon(\tau) = \epsilon_o(\tau) + \frac{\mathcal{N}(\tau)\,\alpha(\tau)}{1 - \theta(\tau)\mathcal{N}(\tau)\,\alpha(\tau)}, \tag{5}$$

with $\theta(\tau) = L(\tau)[1 + \phi(\tau)]/\epsilon_o(\tau)$. Although particle interaction is taken into account in (5), it strictly applies to diluted physical mixtures only, because the underlying dipole approximation (or even a more general multipole expansion) of any real particle is applicable provided that the inter-particle distance is much large than the particle dimensions. This well-known limitation for deterministic media can be extended to any particular realization of a random medium. Thus, the mean inter-particle distance must be sufficiently large relative to the mean particle size *and* the associated standard deviations must be small relative to the respective mean values.

3.2 Dipolarizability

Define the macroscopic dipolarization density A of the composite medium $A = \sum_i \alpha_i$ as the random variable. The associated susceptibility X is

$$X(A) = \frac{A}{1 - \theta A}. \tag{6}$$

The pdf of X can be obtained based on the pdf of A using a variate transformation:

$$f_X(x) = f_A\left[A(X = x)\right] \left| \left(\frac{\mathrm{d}A(X)}{\mathrm{d}X} \right)_{X=x} \right|. \tag{7}$$

The inverse function $A(X) = X/(1 + \theta X)$ exhibits a Gauss normal distribution, yielding:

$$f_X(x) = C \frac{\exp\left[-\left[x/(1+\theta x) - \mu_A\right]^2 / \left(2\sigma_A^2\right) \right]}{\sqrt{2\pi}\ \sigma_A\ \left(1+\theta x\right)^2}, \tag{8}$$

where C is a normalization constant (cfr. infra). Although, strictly, $a, x, \theta \geq 0$ for passive media, we have tacitly assumed here that the range of A and X extends from $-\infty$ to $+\infty$, in order to avoid the use of truncated distributions. This commonly used approximation is justified provided that μ_A and μ_X — both strictly positive — are sufficiently large relative to σ_A and σ_X, respectively. Under this assumption, $f_X(x)$ and $f_A(a)$ have decayed to sufficiently low levels at $x = 0$ and $a = 0$ from their values at $x = \mu_X$ and $a = \mu_A$, so that the contribution of the left tails of the pdfs is negligible in the probability calculations. In the dilute limit ($\mathcal{NV} \ll 1$), $X \approx A$ hence a Gauss normal pdf is then retrieved:

$$f_X(x) = \frac{\exp\left[-(x-\mu_A)^2 / \left(2\sigma_A^2\right)\right]}{\sqrt{2\pi}\ \sigma_A}, \quad (\mathcal{NV} \ll 1). \tag{9}$$

If ergodicity in the weak sense holds for a set of $\mathcal{NV}$ realizations of a material sample, (8) and (9) can be expressed in terms of the statistics of the particle's microscopic polarizability α. Indeed, if the $\alpha_i(\tau_j)$ are themselves distributed according to the same Gauss normal pdf when considered across the realization space $\{\tau_j\}$, for arbitrarily chosen i with $j = 1, \ldots, \mathcal{NV}$, i.e. with identical mean values μ_α and (mean) standard deviations $\sigma_\alpha = \sigma_{\alpha_i}$, then the physical sample of material is at the same time a statistical sample taken from a *single* population of particles. In this case $\mu_A = \mathcal{NV}\mu_\alpha$, $\sigma_A = \sqrt{\mathcal{NV}}\sigma_\alpha$ and $\sigma_{\mu_A} =$

$\sigma_A/\sqrt{\mathcal{NV}} = \sigma_\alpha$. As a consequence, μ_{α_i} and σ_{α_i} can then be estimated from the statistics of the sample distribution of $\sum_i \alpha_i$. Then (9) can be expressed as:

$$f_X(x) = \frac{\exp\left[-\mathcal{NV}\left(\alpha - \mu_\alpha\right)^2 / \left(2\sigma_\alpha\right)^2\right]}{\sqrt{2\pi\mathcal{NV}}\,\sigma_\alpha}. \tag{10}$$

If the α_i have a different pdf then ergodicity in the strict sense may be required to arrive at a corresponding result.

The cumulative distribution function (cdf) $F_X(x)$ associated with (8) is, with $\theta > 0$:

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)\mathrm{d}t \\ &= \begin{cases} \frac{1}{2}\left\{\mathrm{erf}\left[\left(\sqrt{2}\theta\sigma_X\right)^{-1}\left[(1-\theta\mu_X) - (1+\theta x)^{-1}\right]\right]\right. \\ \left. -\mathrm{erf}\left[\left(\sqrt{2}\theta\sigma_X\right)^{-1}(1-\theta\mu_X)\right]\right\}, \quad (x \le -\theta^{-1}); \\ \\ 1 - \frac{1}{2}\left\{\mathrm{erf}\left[\left(\sqrt{2}\theta\sigma_X\right)^{-1}(1-\theta\mu_X)\right] - \right. \\ \left. \mathrm{erf}\left[\left(\sqrt{2}\theta\sigma_X\right)^{-1}\left[(1-\theta\mu_X) - (1+\theta x)^{-1}\right]\right]\right\}, \quad (x > -\theta^{-1}), \end{cases} \end{aligned} \tag{11}$$

where $\mathrm{erf}(t) \triangleq (2/\sqrt{\pi})\int_0^t \exp(-u^2)\mathrm{d}u$ is the standard error function. The normalization constant in (8), defined by the constraint $F_X(+\infty) = 1$, is therefore $C = 1/2$. A η%-confidence interval for X follows as $[F_X^{-1}((1-\eta/100)/2),\ F_X^{-1}((1+\eta/100)/2)]$. Since $X(A)$ is monotonic, the quantiles of $f_X(x)$ can also directly be calculated from those for $f_A(a)$, i.e.:

$$F_X^{-1}((1 \pm \eta/100)/2) = \frac{F_A^{-1}((1 \pm \eta/100)/2)}{1 - \theta\, F_A^{-1}((1 \pm \eta/100)/2)}. \tag{12}$$

The mean return period $\langle T\rangle$ for a chosen threshold level x, relevant to reliability analysis and quality control in sample manufacturing and testing, follows as $\langle T(x)\rangle = [1-F_X(x)]^{-1}$.

The mean, standard deviation, skewness, kurtosis and higher- (nth-) order moments of $f_X(x)$ can be obtained from the moment integrals $\int_{-\infty}^{+\infty} x^n f_X(x)\mathrm{d}x$ or from the moment

generating function $M_X(s) = \int_{-\infty}^{+\infty} \exp(xs) \cdot f_X(x) \mathrm{d}x$, the latter being equivalent to:

$$\begin{aligned} M_{X(A)}(s) &= \int_{-\infty}^{+\infty} \exp[x(a)s] f_A(a) \mathrm{d}a \\ &= \int_{-\infty}^{+\infty} \frac{\exp\left[\frac{as}{1-\theta a} - \frac{(a-\mu_A)^2}{2\sigma_A^2}\right]}{\sqrt{2\pi}\ \sigma_A} \mathrm{d}a. \end{aligned} \tag{13}$$

For relatively small particle concentrations ($\theta \ll 1$), (13) reduces with $\int_{-\infty}^{+\infty} \exp[-(px^2 + qx + r)]\mathrm{d}x = \sqrt{\pi/p}\, \exp[q^2/(4p) - r]$ to

$$M_{X(A)}(s) = \exp\left[\frac{\mu_A s + (\sigma_A^2/2)\, s^2}{1 - 2\theta\sigma_A^2}\right] / \sqrt{1 - 2\theta\sigma_A^2\, s}. \tag{14}$$

In the dilute limit ($\theta \to 0$), this reduces of course to the moment generating function of the Gauss normal distribution for $X = A$. The mean and variance are obtained from (14) as:

$$\mu_X = \left(\frac{\mathrm{d}M_{X(A)}(s)}{\mathrm{d}s}\right)_{s=0} = \mu_A + \theta\, \sigma_A^2, \tag{15}$$

$$\sigma_X^2 = \left(\frac{\mathrm{d}^2 M_{X(A)}(s)}{\mathrm{d}s^2}\right)_{s=0} - \mu_X^2 = \left(1 + 4\theta\mu_A + 2\theta^2\sigma_A^2\right) \sigma_A^2, \tag{16}$$

hence μ_X and σ_X^2 now depend in particular on σ_A^2 and μ_A to first order in θ, respectively. Thus, the uncertainty in the dipolarization density affects the mean dipolarizability. Similarly, the mean dipolarization density affects the uncertainty of the dipolarizability to first order in θ. Provided $X(A)$ is a sufficiently slowly varying function, one can apply the variance theorem to obtain approximations for the mean and variance more readily: for a general function h of n random variables $X_1, \ldots, X_n$:

$$\mu_{h_{X_1,\ldots,X_n}(x_1,\ldots,x_n)} \approx h_{X_1,\ldots,X_n}(\mu_{X_1}, \ldots, \mu_{X_n}), \tag{17}$$

$$\begin{aligned} \sigma^2_{h_{X_1,\ldots,X_n}(x_1,\ldots,x_n)} \approx \sum_{i=1}^{n} \left[\left(\frac{\partial h_{X_1,\ldots,X_n}}{\partial X_i}\right)_M\right]^2 \sigma^2_{X_i} + 2 \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left(\frac{\partial h_{X_1,\ldots,X_n}}{\partial X_i}\right)_M \left(\frac{\partial h_{X_1,\ldots,X_n}}{\partial X_j}\right)_M \cdot \\ \rho_{X_i,X_j} \sigma_{X_i} \sigma_{X_j}, \end{aligned} \tag{18}$$

where $\rho_{X_i,X_j} \triangleq \sigma^2_{X_i,X_j}/(\sigma_{X_i}\sigma_{X_j})$ is the dimensionless correlation coefficient between X_i and X_j and with $M = (\mu_1, \mu_2, \ldots, \mu_n)$. With $h_X = X(A)$, we obtain:

$$\mu_X \approx X(\mu_A) = \frac{\mu_A}{1 - \theta\mu_A}, \tag{19}$$

which fails to show the dependence on σ_A^2 of (15) however, to first order in θ, although

$$\sigma_X^2 \approx \left[\left(\frac{\partial X(A)}{\partial A}\right)_{A=\mu_A}\right]^2 \sigma_A^2 = \frac{\sigma_A^2}{(1-\theta\mu_A)^4} \tag{20}$$

shows the correct first-order dependence for small θ as in (16). If required, the statistics for A can be traced back to the microscopic mean, standard deviation and correlation coefficients for the individual particles by substituting $\mu_A = \sum_i \mu_{\alpha_i}$ and $\sigma_A^2 = \sum_i \sigma_{\alpha_i}^2 + 2\sum_i \sum_{j\neq i} \rho_{\alpha_i,\alpha_j}\sigma_{\alpha_i}\sigma_{\alpha_j}$.

The constitutive parameters of a batch of ζ nominally identical composite specimens, each being an approximate realization of a specified medium, exhibit sample distributions that are based on the underlying statistical pdf of the parent infinite medium, i.e. the medium universe. In particular, extreme-value distributions for X can be derived for the specimens. For example, the pdfs of the sample maximum X_M, sample minimum X_m and sample range $\Delta X \triangleq X_M - X_m$ of the susceptibility for a set of ζ realizations for specified nominal parameters are

$$f_{X_M}(x_M) = \zeta\,[F_X(x = x_M)]^{\zeta-1}\, f_X(x = x_M), \tag{21}$$

$$f_{X_m}(x_m) = \zeta\,[1 - F_X(x = x_m)]^{\zeta-1}\, f_X(x = x_m), \tag{22}$$

$$f_{\Delta X}(\Delta x) = \zeta(\zeta-1)\int_{-\infty}^{+\infty} [F_X(x+\Delta x) - F_X(x)]^{\zeta-2} f_X(x+\Delta x)\, f_X(x)\,\mathrm{d}x, \tag{23}$$

with $f_X(x)$ and $F_X(x)$ given by (8) and (11), respectively. The mean, standard deviation, confidence intervals etc. of X_M, X_m and ΔX follow in the usual manner.

The effect of randomness of the host medium permittivity $\mathcal{E}_o$, particle interaction and depolarization Λ on X via Θ can be analyzed along similar lines, based on $f_X(x) = f_\Theta(a^{-1} - x^{-1})/x^2$. For example, for spherical particles in a random host medium [$\Theta = (3\mathcal{E}_o)^{-1}$],

assuming a Gauss normal pdf for $\mathcal{E}_o$ and neglecting the effect of fluctuations in $\mathcal{E}_o$ on the particles' dipolarization density a to first order, we obtain:

$$f_X(x) = \frac{a^2 \exp\left[-\left[(ax/3)/(x-a) - \mu_{\mathcal{E}_o}\right]^2 / \left(2\sigma^2_{\mathcal{E}_o}\right)\right]}{3\sqrt{2\pi}\ \sigma_{\mathcal{E}_o}\ \ (x-a)^2}. \tag{24}$$

The normalization constant has been omitted in (24), as will be done in subsequent pdfs as well.

3.3 Effective medium parameters

From the knowledge of $f_X(x)$, the pdf of the effective random permittivity $\mathcal{E} = \epsilon_o + X$ with deterministic ϵ_o follows as $f_{\mathcal{E}}(\epsilon) = f_X(\epsilon - \epsilon_o)$. Similarly, for the effective random permeability $\mathcal{U} = \mu_o(1 + X_m)$ knowledge of the pdf of the magnetic susceptibility $X_m = M/H$ yields $f_{\mathcal{U}}(\mu) = f_{X_m}(\mu/\mu_o - 1)/\mu_o$. Based on the knowledge of $f_{\mathcal{U},\mathcal{E}}(\mu, \epsilon)$, medium statistics such as spatial or discrete-temporal auto- and cross-correlations, mean and standard deviation for refractive index, wave number, wave impedance, etc. can be derived. We further assume the composite to be a dilute mixture whence $X \approx A$, $X_m \approx A_m$. Then, following (9), $\mathcal{U}$ and $\mathcal{E}$ are statistically independent Gauss normal random variables (see Section 3.5) and can be centered at their mean values to yield the fluctuations (residuals) $\mathcal{U} - \langle\mathcal{U}\rangle$ and $\mathcal{E} - \langle\mathcal{E}\rangle$. Such will be occasionally assumed in order to simplify the calculations.

Using one of various mixture formulas, the pdf of the effective permittivity $\mathcal{E}_{eff}$ of a two-phase mixture with particle permittivity ϵ_p and volume faction v_p can be calculated in function of $f_{\mathcal{E}_p}(\epsilon_p)$, $f_{\mathcal{E}_o}(\epsilon_o)$ or $f_{\mathcal{V}_p}(v_p)$. For example, relevant to the analysis and synthesis of self-adaptive material systems [24], consider the asymmetric Bruggeman–Hanai formula $(\epsilon_{p,r} - \epsilon_{eff,r})/(\epsilon_{p,r} - 1) = (1 - v_p)\epsilon_{eff,r}^{1/3}$ i.e.:

$$\begin{aligned} \epsilon_{eff,r} \;\; = \;\; & \epsilon_{p,r} - (1 - v_p)(\epsilon_{p,r} - 1) \cdot \\ & \left\{ \left[\sqrt{\left(\frac{\epsilon_{p,r}}{2}\right)^2 + \left[\frac{(1 - v_p)(\epsilon_{p,r} - 1)}{3}\right]^3} + \frac{\epsilon_{p,r}}{2} \right]^{\frac{1}{3}} \end{aligned}$$

$$- \left[\sqrt{\left(\frac{\epsilon_{p,r}}{2}\right)^2 + \left[\frac{(1-v_p)(\epsilon_{p,r}-1)}{3}\right]^3} - \frac{\epsilon_{p,r}}{2} \right]^{\frac{1}{3}} \Bigg\} \tag{25}$$

where $\epsilon_{\alpha,r} \triangleq \epsilon_\alpha/\epsilon_o$. For a given $f_{\mathcal{V}_p}(v_p)$, $f_{\mathcal{E}_{eff,r}}(\epsilon_{eff,r}) = f_{\mathcal{V}_p}[v_p(\epsilon_{eff,r})]\ |(\mathrm{d}\mathcal{V}_p/\mathrm{d}\mathcal{E}_{eff,r})_{\epsilon_{eff,r}}|$ follows with (25) as:

$$f_{\mathcal{E}_{eff,r}}(\epsilon_{eff,r}) = f_{\mathcal{V}_p}\left(1 - \frac{\epsilon_{eff,r}^{1/3}(\epsilon_{p,r}-\epsilon_{eff,r})}{\epsilon_{p,r}-1}\right) \frac{4\epsilon_{eff,r}^{1/3} - \epsilon_{p,r}}{3(\epsilon_{p,r}-1)}. \tag{26}$$

Likewise, the effect of a pdf and uncertainty for $\mathcal{E}_{p,r}$ due to particle manufacturing, shape, surface effect etc. on $f_{\mathcal{E}_{eff,r}}(\epsilon_{eff,r})$ can be studied. The statistics of other mixing rules are analyzed along similar lines. Similar expressions for the effective permeability follow from duality.

3.4 Loss tangent, quality factor, dispersion

The random dielectric loss tangent $\mathrm{Tan}\,\Delta \triangleq \mathcal{E}''/\mathcal{E}'$ or, equivalently, the random quality factor of the medium $Q \triangleq (\mathrm{Tan}\,\Delta)^{-1}$ is the ratio of two Gauss normally distributed random variables for which we obtain, using (3.462.1) and (9.254.2) of [25]:

$$\begin{aligned} f_{\mathrm{Tan}\,\Delta}(\tan\delta) &= \int_{-\infty}^{+\infty} |u| f_{\mathcal{E}''}(u\tan\delta) f_{\mathcal{E}'}(u)\mathrm{d}u \qquad (27)\\ &= \frac{\exp\left[-\left[(\mu_{\mathcal{E}'}/\sigma_{\mathcal{E}'})^2 + (\mu_{\mathcal{E}''}/\sigma_{\mathcal{E}''})^2\right]/2\right]}{2\sigma_{\mathcal{E}'}\sigma_{\mathcal{E}''}\left[(1/\sigma_{\mathcal{E}'})^2 + (\tan\delta/\sigma_{\mathcal{E}''})^2\right]} \cdot \\ &\quad \Bigg\{ 1 + \sqrt{\frac{2}{\pi}} \frac{\mu_{\mathcal{E}'}/\sigma_{\mathcal{E}'}^2 + (\tan\delta)\mu_{\mathcal{E}''}/\sigma_{\mathcal{E}''}^2}{\sqrt{(1/\sigma_{\mathcal{E}'})^2 + (\tan\delta/\sigma_{\mathcal{E}''})^2}} \cdot \\ &\quad\quad \mathrm{erf}\left[\frac{\mu_{\mathcal{E}'}/\sigma_{\mathcal{E}'}^2 + (\tan\delta)\mu_{\mathcal{E}''}/\sigma_{\mathcal{E}''}^2}{\sqrt{2}\sqrt{(1/\sigma_{\mathcal{E}'})^2 + (\tan\delta/\sigma_{\mathcal{E}''})^2}}\right] \cdot \\ &\quad\quad \exp\left[\frac{\left[\mu_{\mathcal{E}'}/\sigma_{\mathcal{E}'}^2 + (\tan\delta)\mu_{\mathcal{E}''}/\sigma_{\mathcal{E}''}^2\right]^2}{2\left[(1/\sigma_{\mathcal{E}'})^2 + (\tan\delta/\sigma_{\mathcal{E}''})^2\right]}\right] \Bigg\}. \qquad (28) \end{aligned}$$

For standardized circular Gauss normal $\mathcal{E}$, this reduces to the standard Cauchy–Breit–Wigner distribution $f_{\mathrm{Tan}\,\Delta}(\tan\delta) = [\pi\,(1+\tan^2\delta)]^{-1}$. Note, however, that $\mathcal{E}'$ and $\mathcal{E}''$ are connected via the Kramers–Kronig relations and are therefore not strictly independent as is

required for the application of (27). Therefore (28) only serves as an approximation, valid within a sufficiently narrow bandwidth of operation where $\mathcal{E}'$ and $\mathcal{E}''$ can be considered as approximately independent. Alternatively, if the cross-correlation $\rho_{\mathcal{E}',\mathcal{E}''}$ and, hence, the joint pdf $f_{\mathcal{E}',\mathcal{E}''}$ is known then $\mathrm{Tan}\,\Delta$ can be calculated as a marginal distribution. The pdf $f_Q(q)$ of Q is of course obtained by interchanging the subscripts $\mathcal{E}'$ and $\mathcal{E}''$, and replacing $\mathrm{Tan}\,\Delta$ and $\tan\delta$ by Q and q in (28).

From $\mathrm{Tan}\,\Delta = (\epsilon'_s - \epsilon'_\infty)\omega\mathcal{T}/[\epsilon'_s + (\omega\mathcal{T})^2\epsilon'_\infty]$ for dielectrics exhibiting first-order (Debye) relaxation, the pdf of the relaxation time $\mathcal{T}$ is obtained as:

$$f_{\mathcal{T}}(\tau) = f_{\mathrm{Tan}\,\Delta}\left(\frac{\omega\tau(\epsilon'_s - \epsilon'_\infty)}{\epsilon'_s + (\omega\tau)^2\epsilon'_\infty}\right)\frac{\omega(\epsilon'_s - \epsilon'_\infty)|\epsilon'_s - (\omega\tau)^2\epsilon'_\infty|}{[\epsilon'_s + (\omega\tau)^2\epsilon'_\infty]^2}, \tag{29}$$

with $f_{\mathrm{Tan}\,\Delta}(\cdot)$ given by (28), from which the pdf of the relaxation center frequency $\Omega_c = \mathcal{T}^{-1}$ follows as $f_{\Omega_c}(\omega_c) = f_{\mathcal{T}}(\Omega_c^{-1})/\Omega_c^2$. In the Cole–Davidson equation $\epsilon = \epsilon'_\infty + (\epsilon'_s - \epsilon'_\infty)/[1 + j(\omega\tau)^{1-\beta}]$, the temperature-dependent dimensionless parameter β accounts for the broadening of the relaxation spectrum in solids and liquids ($0 < \beta \leq 1$) as compared to relaxation in dilute gases for which $\beta = 0$ [26]. For example, based on the same physical assumptions as those for the distribution of A, if β becomes random with a Gauss normal pdf centered at $\mu_{\mathcal{B}}$, then $f_{\mathcal{E}'}(\epsilon')$ and $f_{\mathcal{E}''}(\epsilon'')$ are found as:

$$\begin{aligned} f_{\mathcal{E}'}(\epsilon') &= \left[2\sqrt{2\pi}\,\sigma_{\mathcal{B}}\,(\epsilon'_s - \epsilon'_\infty)\,|\ln(\omega\tau)|\,x(1-x)\right]^{-1}\cdot \\ &\qquad \exp\left\{-\frac{[1 - \ln(x^{-1}-1)/[2\ln(\omega\tau)] - \mu_{\mathcal{B}}]^2}{2\sigma_{\mathcal{B}}^2}\right\}, \end{aligned} \tag{30}$$

$$\begin{aligned} f_{\mathcal{E}''}(\epsilon'') &= \frac{(2y)^{-2}}{\sqrt{2\pi}\,\sigma_{\mathcal{B}}\,(\epsilon'_s - \epsilon'_\infty)\,|\ln(\omega\tau)|\,\sqrt{(2y)^{-2} - 1}}\cdot \\ &\qquad \exp\left\{-\frac{\left[1 - \cosh^{-1}[(2y)^{-1}]/\ln(\omega\tau) - \mu_{\mathcal{B}}\right]^2}{2\sigma_{\mathcal{B}}^2}\right\}, \end{aligned} \tag{31}$$

in which $x \triangleq (\epsilon' - \epsilon'_\infty)/(\epsilon'_s - \epsilon'_\infty)$ and $y \triangleq \epsilon''/(\epsilon'_s - \epsilon'_\infty)$. The pdf of Tan Δ due to uncertainties in β can then be calculated as $f_{\mathrm{Tan}\,\Delta}[\tan\delta(\beta)] = \int_{-\infty}^{+\infty}|u| f_{\mathcal{E}''}[u\tan\delta(\beta)] f_{\mathcal{E}'}(u)\mathrm{d}u$ to complement (28). Of practical importance is the inverse operation, viz. the com-

putation of the pdf of the spectral broadening exponent $\mathcal{B}$ from measurement of effective medium statistics and distributions. For example, for use with (28) we obtain for $f_{\mathcal{B}}(\beta) = f_{\mathrm{Tan}\,\Delta}\left(\tan\delta(\beta)\right)\left|(\mathrm{d}[\mathrm{Tan}\Delta(\mathcal{B})]/\mathrm{d}\mathcal{B})_{\mathcal{B}=\beta}\right|$:

$$\begin{aligned} f_{\mathcal{B}}(\beta) &= f_{\mathrm{Tan}\,\Delta}\left(\frac{(\omega\tau)^{1-\beta}\left(\epsilon'_s-\epsilon'_\infty\right)}{\epsilon'_s+(\omega\tau)^{2(1-\beta)}\,\epsilon'_\infty}\right)\cdot \\ &\quad \frac{(\omega\tau)^{1-\beta}\left(\epsilon'_s-\epsilon'_\infty\right)\left|\ln(\omega\tau)\right|\left|\epsilon'_s-(\omega\tau)^{2(1-\beta)}\epsilon'_\infty\right|}{\left[\epsilon'_s+(\omega\tau)^{2(1-\beta)}\epsilon'_\infty\right]^2}. \end{aligned} \tag{32}$$

The statistical analysis of Williams–Watts and Havriliak–Negami relaxation formulations can be performed along similar lines.

Second-order (Lorentz) resonance absorption, while confined to infrared and optical wavelengths for homogeneous dielectrics, is relevant to composites where the ambient wavelength is of the order of the characteristic length of the particles, hence often occurring at much lower frequencies. These media are characterized by a static permittivity ϵ_s, plasma frequency ω_p, absorption bandwidth a and resonance frequency ω_c. The relative permittivity $\epsilon/\epsilon_o = 1+\omega_p^2/(\omega_c^2-\omega^2+j\omega a)$ becomes for quasi-static excitation $\epsilon/\epsilon_o \approx (1+\omega_p^2/\omega_c^2) - ja\omega_p^2\omega/\omega_c^4$, to first order in (ω/ω_c), and its pdf can be calculated as above. For example, in the case of a random resonance frequency Ω_c:

$$f_{\mathcal{E}'}(\epsilon') = \frac{1}{2\epsilon_o}\left(\frac{\omega_p^2}{(\epsilon'/\epsilon_o-1)^3}\right)^{\frac{1}{2}} f_{\Omega_c}\left[\left(\frac{\omega_p^2}{\epsilon'/\epsilon_o-1}\right)^{\frac{1}{2}}\right], \tag{33}$$

$$f_{\mathcal{E}''}(\epsilon'') = \frac{1}{4}\left(\frac{\epsilon_o a\omega_p^2\omega}{\epsilon''^5}\right)^{\frac{1}{4}} f_{\Omega_c}\left[\left(\frac{\epsilon_o a\omega_p^2\omega}{\epsilon''}\right)^{\frac{1}{4}}\right]. \tag{34}$$

Conversely, $f_{\Omega_c}(\omega_c)$ can be deduced from $f_{\mathcal{E}'}(\epsilon')$ or $f_{\mathcal{E}''}(\epsilon'')$ as

$$\begin{aligned} f_{\Omega_c}(\omega_c) &= (2\epsilon_o\omega_p^2/\omega_c^3)\, f_{\mathcal{E}'}\left[\epsilon_o(1+\epsilon_o\omega_p^2/\omega_c^2)\right] \\ &= (4\epsilon_o a\omega_p^2\omega/\omega_c^5)\, f_{\mathcal{E}''}(\epsilon_o a\omega_p^2\omega/\omega_c^4). \end{aligned} \tag{35}$$

The significance of the latter result is that the statistics of Ω_c can be estimated from a knowledge of $f_{\mathcal{E}'^{(\prime)}}(\epsilon'^{(\prime)})$ well outside the resonance band.

3.5 Refractive index, wave number, intrinsic impedance

The importance of the concepts of wave number and wave impedance in electrical engineering applications is well appreciated. In addition, the wave number and wave impedance are sometimes the preferred material reference parameters for plane-wave excitation, avoiding ambiguities between different constitutive formalisms when a chosen set of conventional constitutive parameters is not statistically independent, e.g. when magnetoelectric coupling occurs [27].

For centralized $\mathcal{U}$ and $\mathcal{E}$, the pdf of the refractive index $N = \sqrt{\mathcal{UE}/(\mu_o\epsilon_o)}$ is obtained after subsequent variate transformations $M = \mathcal{UE}$ and $N = \sqrt{M/(\mu_o\epsilon_o)}$. Using (3.471.12) of [25]:

$$\begin{aligned} f_M(m) &= \int_{-\infty}^{+\infty} |u|^{-1} f_{\mathcal{U}}(u) f_{\mathcal{E}}(m/u)\mathrm{d}u \\ &= (2\pi\sigma_{\mathcal{U}}\sigma_{\mathcal{E}})^{-1} \int_{-\infty}^{+\infty} |u|^{-1} \exp\left[-\frac{u^2}{2\sigma_{\mathcal{U}}^2} - \frac{(m/u)^2}{2\sigma_{\mathcal{E}}^2}\right] \mathrm{d}u \\ &= (\pi\sigma_{\mathcal{U}}\sigma_{\mathcal{E}})^{-1} K_0\left[\frac{|m|}{\sigma_{\mathcal{U}}\sigma_{\mathcal{E}}}\right] \end{aligned} \tag{36}$$

where $K_0(\cdot)$ is the modified Bessel function of the second kind of order zero. With $M = N^2\mu_o\epsilon_o$, we obtain after transformation and normalization:

$$f_N(n) = \frac{2\mu_o\epsilon_o|n|}{\pi\sigma_{\mathcal{U}}\sigma_{\mathcal{E}}} K_0\left[\frac{\mu_o\epsilon_o n^2}{\sigma_{\mathcal{U}}\sigma_{\mathcal{E}}}\right]. \tag{37}$$

Using (6.521.2) of [25], the standard deviation follows as:

$$\sigma_N = \sqrt{(2\sigma_{\mathcal{U}}\sigma_{\mathcal{E}}) \,/\, (\pi\mu_o\epsilon_o)}. \tag{38}$$

If N is a sufficiently slowly varying function of $\mathcal{U}$ and $\mathcal{E}$, then (18) with $h_{\mathcal{U},\mathcal{E}} = \sqrt{\mathcal{UE}/(\mu_o\epsilon_o)}$ yields:

$$\sigma_N^2 \approx \frac{\mu_{\mathcal{E}}}{4n_o^2\mu_{\mathcal{U}}}\sigma_{\mathcal{U}}^2 + \frac{\mu_{\mathcal{U}}}{4n_o^2\mu_{\mathcal{E}}}\sigma_{\mathcal{E}}^2 + \frac{\rho_{\mathcal{U},\mathcal{E}}}{2n_o^2}\sigma_{\mathcal{U}}\,\sigma_{\mathcal{E}}, \tag{39}$$

where $n_o \triangleq v_o^{-1} = \sqrt{\mu_o\epsilon_o}$. For $\mu_{\mathcal{U}}, \mu_{\mathcal{E}} \to 0$, (39) tends to $(\sigma_{\mathcal{U}}^2 + \sigma_{\mathcal{E}}^2 + 2\rho_{\mathcal{U},\mathcal{E}}\sigma_{\mathcal{U}}\sigma_{\mathcal{E}})/(4n_o^2)$. This is a more general though approximate result compared to (38), because (36)–(38)

assume $\mathcal{U}$ and $\mathcal{E}$ to be statistically independent. Furthermore, $\mu_N \approx \sqrt{\mu_{\mathcal{U}}\mu_{\mathcal{E}}} = 0$. If the composite has no net (macroscopic) curvature in its microstructure or no net curl in its macroscopic field, so that for the local random electric current density $\underline{J}(\underline{j})$ at $\underline{r}$ the associated magnetic moment $(1/2)\int_{L_i}[\underline{r} \times \underline{J}(\underline{r})]\mathrm{d}l$ vanishes with probability one, then electrical currents do not give rise to net induced magnetic moments, hence $\rho_{\mathcal{U},\mathcal{E}} = 0$. However, for certain complex composite media $\rho_{\mathcal{U},\mathcal{E}} \neq 0$, e.g. composites which exhibit ferromagneto-ferroelectric coupling [28]. If the value of $\rho_{\mathcal{U},\mathcal{E}}$ is not known, but in any case nonzero, then one replaces σ_N in (39) by its upper limit:

$$\sigma_N \leq \left[\sqrt{\frac{\mu_{\mathcal{E}}}{\mu_{\mathcal{U}}}}\sigma_{\mathcal{U}} + \sqrt{\frac{\mu_{\mathcal{U}}}{\mu_{\mathcal{E}}}}\sigma_{\mathcal{E}}\right]/(2n_o). \tag{40}$$

For a pure dielectric medium ($\sigma_{\mathcal{U}} = 0$), (39) shows that $\nu_N \approx \nu_{\mathcal{E}}\sqrt{\mu_{\mathcal{U}}\mu_{\mathcal{E}}}/(2n_o)$, where $\nu \triangleq \sigma/\mu$.

The mean-centralized wave number $K = \Omega\sqrt{\mathcal{U}\mathcal{E}}$ has pdf

$$f_K(k) = k_o^{-1} f_N(k/k_o), \tag{41}$$

where $k_o \triangleq \omega n_o$, if the angular frequency Ω is deterministic with value ω. If Ω is itself random, mean-centralized and independent of $\sqrt{\mathcal{U}\mathcal{E}}$, as in the case of a dispersionless medium, then $f_K(k) = \int_{-\infty}^{+\infty}(n_o|u|)^{-1}\cdot\, f_\Omega(k/u)f_N(u/n_o)\mathrm{d}u$. This occurs, for example, when generating complex field distributions in a mode-stirred or mode-tuned reverberation chamber. We defer this analysis and discussion to Section 5.3.

For the intrinsic impedance $Z = \sqrt{\mathcal{U}/\mathcal{E}}$, the subsequent variate transformations $M = \mathcal{U}/\mathcal{E}$ and $Z = \sqrt{M}$ can be applied. The former yields (28) but with $\mathrm{Tan}\,\Delta$, $\mathcal{E}'$ and $\mathcal{E}''$ replaced by M, $\mathcal{E}$ and $\mathcal{U}$, respectively. With $f_Z(z) = f_M[M(z)]|(\mathrm{d}M(Z)/\mathrm{d}Z)_{Z=z}|$ we arrive at:

$$\begin{aligned} f_Z(z) &= 2|z|\int_{-\infty}^{+\infty}|u|f_{\mathcal{U}}(uz)f_{\mathcal{E}}(u)\mathrm{d}u \\ &= \frac{|z|\,\exp\left[-\left[(\mu_{\mathcal{E}}/\sigma_{\mathcal{E}})^2 + (\mu_{\mathcal{U}}/\sigma_{\mathcal{U}})^2\right]/2\right]}{\sigma_{\mathcal{E}}\sigma_{\mathcal{U}}\left[(1/\sigma_{\mathcal{E}})^2 + (z/\sigma_{\mathcal{U}})^2\right]}\,. \end{aligned} \tag{42}$$

$$\left\{1+\sqrt{\frac{2}{\pi}}\frac{\mu_{\mathcal{E}}/\sigma_{\mathcal{E}}^2+z\mu_{\mathcal{U}}/\sigma_{\mathcal{U}}^2}{\sqrt{\left(1/\sigma_{\mathcal{E}}\right)^2+\left(z/\sigma_{\mathcal{U}}\right)^2}}\cdot\right.$$
$$\operatorname{erf}\left[\frac{\mu_{\mathcal{E}}/\sigma_{\mathcal{E}}^2+z\mu_{\mathcal{U}}/\sigma_{\mathcal{U}}^2}{\sqrt{2}\sqrt{\left(1/\sigma_{\mathcal{E}}\right)^2+\left(z/\sigma_{\mathcal{U}}\right)^2}}\right]\cdot$$
$$\left.\exp\left[\frac{\left[\mu_{\mathcal{E}}/\sigma_{\mathcal{E}}^2+z\mu_{\mathcal{U}}/\sigma_{\mathcal{U}}^2\right]^2}{2\left[\left(1/\sigma_{\mathcal{E}}\right)^2+\left(z/\sigma_{\mathcal{U}}\right)^2\right]}\right]\right\}. \tag{43}$$

For mean-centralized $\mathcal{U}$ and $\mathcal{E}$, this reduces after normalization to:

$$f_Z(z)=\frac{(2/\pi)\ |z|}{(\sigma_{\mathcal{U}}/\sigma_{\mathcal{E}})+(\sigma_{\mathcal{E}}/\sigma_{\mathcal{U}})z^2}. \tag{44}$$

It has zero mean and although the standard deviation is indeterminate, (44) suggests the scale parameter $\sqrt{\sigma_{\mathcal{U}}/\sigma_{\mathcal{E}}}$ as a measure for the dispersion. Application of (18) yields:

$$\sigma_Z^2\approx\frac{1}{4\mu_{\mathcal{U}}\mu_{\mathcal{E}}}\sigma_{\mathcal{U}}^2+\frac{4\mu_{\mathcal{U}}}{9\mu_{\mathcal{E}}^3}\sigma_{\mathcal{E}}^2-\frac{2\rho_{\mathcal{U},\mathcal{E}}}{3\mu_{\mathcal{E}}^2}\sigma_{\mathcal{U}}\,\sigma_{\mathcal{E}}, \tag{45}$$

whereas $\mu_Z\approx\sqrt{\mu_{\mathcal{U}}/\mu_{\mathcal{E}}}$. The upper limit of σ_Z is $\sqrt{\mu_{\mathcal{U}}/\mu_{\mathcal{E}}}[\nu_{\mathcal{U}}/2+(2/3)\nu_{\mathcal{E}}]$. For a pure dielectric, $\nu_Z\approx(2/3)\nu_{\mathcal{E}}$. The difference in sign for the last term in (39) and (45) shows that nonzero correlation between the random permittivity and random permeability gives rise to an opposing trend for σ_N^2 as compared to the effect on σ_Z^2.

3.6 Dyadic Green function

The pdf of the scalar Green function $G(\underline{r}|\underline{r}')=\exp(-jK|\underline{r}-\underline{r}'|)/(4\pi|\underline{r}-\underline{r}'|)$ follows for random K as:

$$f_{G(\underline{r}|\underline{r}')}[g(\underline{r}|\underline{r}')]=f_K\left(\frac{j\ln[4\pi\,|\underline{r}-\underline{r}'|\,g(\underline{r}|\underline{r}')]}{|\underline{r}-\underline{r}'|}\right)\left[|\underline{r}-\underline{r}'|\,g(\underline{r}|\underline{r}')\right]^{-1}, \tag{46}$$

from which the pdf of the dyadic Green function of the infinite homogenized medium can be deduced, e.g. for $\underline{\underline{G}}(\underline{r}|\underline{r}')=(\underline{\underline{I}}+K^{-2}\underline{\nabla}\,\underline{\nabla})\,G(\underline{r}|\underline{r}')$.

For a specimen with a finite number of particles, the $\mathcal{NV}$-particle Green dyadic for the

multiple-scattering problem satisfies a dyadic linear Fredholm integral equation of the second kind in $\underline{\underline{G}}$ (see e.g. [29, Section 8.1], [30, Section 6.3.2]), viz.

$$\underline{\underline{G}}(\underline{r}|\underline{r}') - \int\int\int_{\mathcal{V}} \left[\sum_{l=1}^{\mathcal{N}\mathcal{V}} \underline{\underline{G}}_0(\underline{r}|\underline{r}'') \cdot \underline{\underline{Q}}_l(\underline{K}_l, \underline{K}_0; \underline{r}'') \cdot \delta(\underline{r}'' - \underline{r}_l'') \right] \cdot \underline{\underline{G}}(\underline{r}''|\underline{r}') \, \mathrm{d}\underline{r}'' = \underline{\underline{G}}_0(\underline{r}|\underline{r}'), \tag{47}$$

where $\underline{r}_{(l)}''$ corresponds to a source point (inside particle l), $\underline{\underline{G}}_0(\underline{r}|\underline{r}')$ is the Green dyadic of the background medium, and $\underline{K}_l$ and $\underline{K}_0$ are the random wave vectors in particle l and host medium 0, respectively. The integration is effectively over the particle volumes $\mathcal{V}_l$. The mean of the mass kernel, $\underline{\underline{Q}} \triangleq \langle \underline{\underline{Q}}_l(\underline{K}_l, \underline{K}_0; \underline{r}'') \rangle$, is proportional to the mean polarizability of the effective medium with spatial dispersion:

$$\underline{\underline{Q}} = |\underline{K}_0|^2 X/\mathcal{E}_0 = |\underline{K}_0|^2 [\underline{\underline{\mathcal{E}}}(\underline{r}, \underline{r}'')/\mathcal{E}_0 - \underline{\underline{I}}]. \tag{48}$$

For reciprocal composites, i.e. $\underline{\underline{G}}(\underline{r}|\underline{r}') = \underline{\underline{G}}^T(\underline{r}'|\underline{r})$, and symmetric kernel in (47), the equation can be solved for $\underline{\underline{G}}(\underline{r}|\underline{r}')$ from the associated eigenvalue problem – analytically, by converting the integral equation into a differential equation, or numerically, through matrix inversion [31]. The explicit solution of (47) for $\underline{\underline{G}}(\underline{r}|\underline{r}')$ is outside the scope of the present paper, but writing (47) in operator form as $(\mathcal{I} - \mathcal{L}_0)\underline{\underline{G}}(\underline{r}|\underline{r}') = \underline{\underline{G}}_0(\underline{r}|\underline{r}')$, where $\mathcal{I}$ is the identity operator and

$$\mathcal{L}_0 \, \underline{\underline{G}}(\underline{r}|\underline{r}') \triangleq \int\int\int_{\mathcal{V}} \underline{\underline{K}}_0(\underline{r}, \underline{r}'') \cdot \underline{\underline{G}}(\underline{r}''|\underline{r}') \, \mathrm{d}\underline{r}'' \tag{49}$$

with

$$\underline{\underline{K}}_0(\underline{r}, \underline{r}'') \triangleq \sum_{l=1}^{\mathcal{N}\mathcal{V}} \underline{\underline{G}}_0(\underline{r}|\underline{r}'') \cdot \underline{\underline{Q}}_l(\underline{K}_l, \underline{K}_0; \underline{r}'') \delta(\underline{r}'' - \underline{r}_l''), \tag{50}$$

we obtain, for nonsingular $\mathcal{I} - \mathcal{L}_0$:

$$\underline{\underline{G}}(\underline{r}|\underline{r}') = (\mathcal{I} - \mathcal{L}_0)^{-1} \underline{\underline{G}}_0(\underline{r}|\underline{r}') = \sum_{n=0}^{+\infty} \mathcal{L}_0^n \underline{\underline{G}}_0(\underline{r}|\underline{r}'), \tag{51}$$

in which the nth term acccounts for multiple scattering of order n. An alternative solution technique, for infinite media and Gauss normal $\underline{\underline{Q}}_l$, is the perturbation expansion resulting

in the Dyson equation for the mean Green dyadic $\langle \underline{\underline{G}} \rangle$ and the Bethe–Salpeter equation for the covariance Green tetradic $\langle \underline{\underline{G}}^{\dagger}\, \underline{\underline{G}} \rangle$. Closed-form analytical results are only obtainable for certain approximations and assumptions for the effective field in (47). For the mean and covariance, the equivalent of (47) can be solved in the nonlinear Bourret bilocal approximation for $\underline{\underline{Q}}$ [29, 30, 32].

Since (47) can also be conceived as a Fredholm equation of the second kind for $\underline{\underline{G}}_0(\underline{r}|\underline{r}')$, the inverse problem $\underline{\underline{G}}_0(\underline{\underline{G}})$ for a random host medium can be solved in a similar manner, whence the dyadic pdf of $\underline{\underline{G}}(\underline{r}|\underline{r}')$ for an assumed or known pdf of $\underline{\underline{G}}_0(\underline{r}|\underline{r}')$ follows as:

$$f_{\underline{\underline{G}}}(\underline{\underline{g}}) = f_{\underline{\underline{G}}_0}\left[\underline{\underline{G}}_0\left(\underline{\underline{g}}\right)\right]\ |\mathcal{J}(\underline{\underline{g}})|, \tag{52}$$

as defined by

$$\underline{\underline{G}}_0(\underline{\underline{g}}) = \sum_{n=0}^{+\infty}(-1)^n[\mathcal{L}^n\underline{\underline{G}}(\underline{r}|\underline{r}')]_{\underline{\underline{G}}=\underline{\underline{g}}} \tag{53}$$

with

$$\mathcal{L}\,\underline{\underline{G}}(\underline{r}|\underline{r}') = \int\int\int_{\mathcal{V}} \underline{\underline{G}}(\underline{r}|\underline{r}'') \cdot \underline{\underline{K}}(\underline{r}', \underline{r}'')\mathrm{d}\underline{r}'', \tag{54}$$

$$\underline{\underline{K}}(\underline{r}', \underline{r}'') = \sum_{l=1}^{\mathcal{N}\mathcal{V}} \underline{\underline{Q}}_l(\underline{K}_l, \underline{K}_0; \underline{r}'') \cdot \underline{\underline{G}}(\underline{r}''|\underline{r}')\delta(\underline{r}'' - \underline{r}_l''), \tag{55}$$

and where

$$\mathcal{J}(\underline{\underline{g}}) = \det[\partial G_{0_{ij}}/\partial G_{kl}]_{g_{kl}} \tag{56}$$

is the jacobian of the inverse transformation with $\underline{\underline{G}}_0 \equiv [G_{0_{ij}}]$ and $\underline{\underline{G}} \equiv [G_{kl}]$.

Accepting the fact that an exact expression for $\underline{\underline{Q}}$ is unavailable [29, 30], even in the deterministic case, it appears sensible to consider $\underline{\underline{Q}}$ itself as a random variable with associated uncertainty and pdf $f_{\underline{\underline{Q}}}(\underline{\underline{q}})$ in order to obtain realistic and useful solutions. From

the Dyson equation for $\langle \underline{\underline{G}} \rangle$, which is a Fredholm integral equation of the first kind in $\underline{\underline{Q}}$, the random Green function in integral operator form follows as $\mathcal{G} = \mathcal{G}_0(\mathcal{I} + \mathcal{Q}\mathcal{G})$, hence:

$$\mathcal{Q}(\mathcal{G}) = \mathcal{G}_0^{-1}(\mathcal{G} - \mathcal{G}_0)\mathcal{G}^{-1}, \tag{57}$$

where

$$\mathcal{L}\underline{\underline{\varphi}} \triangleq \int\int\int_{\mathcal{V}} \underline{\underline{L}}(\underline{r}|\underline{r}'') \cdot \underline{\underline{\varphi}}(\underline{r}''|\underline{r}')\mathrm{d}\underline{r}'' \tag{58}$$

for $\mathcal{L} \triangleq \mathcal{Q}, \mathcal{G}_0$ and $\mathcal{G}^{(-1)}\underline{\underline{I}} = \langle \underline{\underline{G}} \rangle^{(-1)}$. The pdf of $\langle \underline{\underline{G}} \rangle$ is therefore:

$$f_{\langle \underline{\underline{G}} \rangle}(\underline{\underline{g}}) = f_{\underline{\underline{Q}}}\left[\underline{\underline{Q}}\left(\underline{\underline{g}}\right)\right] \ |\mathcal{J}(\underline{\underline{g}})|, \tag{59}$$

now with

$$\mathcal{J}(\underline{\underline{g}}) = \det[\partial Q_{ij}/\partial\langle G_{kl}\rangle]_{g_{kl}}. \tag{60}$$

4 Random fields

4.1 Energy density

A statistically isotropic, homogeneous and unpolarized complex electric field $\underline{E}$ has three random and statistically independent complex Cartesian components E_α, which in turn are independent of each one of the magnetic field components H_β $(\alpha, \beta = 1, 2, 3)$. For each Cartesian component, the in-phase and quadrature components, E'_α and E''_α, together exhibit a circular Gauss normal distribution $f_{E_\alpha}(e_\alpha) \triangleq f_{E'_\alpha}(e'_\alpha) - j f_{E''_\alpha}(e''_\alpha)$ with variance σ^2 and zero mean per component. Hence, the squared magnitude of each Cartesian complex field component exhibits a χ^2 distribution with two degrees of freedom (χ^2_2), being the sum of squares of two independent identically distributed Gauss normal random variables. If the medium is deterministic, then the associated local energy density $S_\alpha = (\epsilon|E_\alpha|^2 + \mu|H_\beta|^2)/2$ has the same distribution. To obtain the energy density $S_t = (\epsilon|E_t|^2 + \mu|H_t|^2)/2$ associated with the total field, the energy densities of the three Cartesian components are to be added,

resulting in a χ^2_6 distribution for S_t. These χ^2_{2p} pdfs can be written in generic form as

$$f_S(s) = s^{p-1} \exp\left[-s/(2\sigma^2)\right]/[(p-1)!\,(2\sigma^2)^p], \tag{61}$$

with $p = 1$ for a Cartesian component and $p = 3$ for the total-field rms energy density. This standard form emphasizes the relationship with the underlying Gauss normal pdf, through the use of the variance σ^2 of the latter as a parameter of the pdf [33]. Alternatively, the pdf can be rewritten in self-sufficient form using the variance $\sigma^2_{\chi^2_{2p}}$ of the χ^2_{2p} distribution as [34]:

$$f_S(s) = \frac{p^{p/2}}{(p-1)!\,\sigma^p_{\chi^2_{2p}}}\, s^{p-1} \exp\left(-\sqrt{p}\; s/\sigma_{\chi^2_{2p}}\right), \tag{62}$$

where $\sigma^2_{\chi^2_{2p}} = 2(2p)\sigma^4$. Specifically, the χ^2_2 or exponential pdf

$$f_{S_\alpha}(s_\alpha) = \exp(-s_\alpha/\sigma_{\chi^2_2})/\sigma_{\chi^2_2} \tag{63}$$

for the energy density associated with a single local Cartesian field component α, and the χ^2_6 pdf

$$f_{S_t}(s_t) = (3\sqrt{3}/2)\; (s_t^2/\sigma^3_{\chi^2_6})\; \exp(-\sqrt{3}s_t/\sigma_{\chi^2_6}) \tag{64}$$

for the energy density associated with the local total field are of practical importance [7].

4.2 Field magnitude

The pdf of the field magnitude (signal envelope) follows easily from the pdf of the energy density via the variate transformation $|E| = \sqrt{|E|^2}$. With $\mathrm{d}|E| = \mathrm{d}|E|^2/(2\sqrt{|E|^2})$ and $f_{|E|}(|e|) = f_{|E|^2}(|e|^2)\; (\mathrm{d}|E|^2/\mathrm{d}|E|)$, we obtain

$$f_{|E|}(|e|) = 2|e|^{2p-1}\; \exp[-|e|^2/(2\sigma^2)]\; /\; [(p-1)!(2\sigma^2)^p], \tag{65}$$

which is a χ distribution with $2p$ degrees of freedom (χ_{2p}). In terms of the variance $\sigma^2_{\chi_{2p}}$ of the distribution itself, this pdf can be rewritten as:

$$f_{|E|}(|e|) \;=\; \frac{2\left[p - \left(\frac{\Gamma(p+1/2)}{\Gamma(p)}\right)^2\right]^p}{(p-1)!\,\sigma^{2p}_{\chi_{2p}}} |e|^{2p-1}\; \exp\left\{-\left[p - \left(\frac{\Gamma(p+1/2)}{\Gamma(p)}\right)^2\right] |e|^2/\sigma^2_{\chi_{2p}}\right\}, \tag{66}$$

where $\Gamma(\cdot)$ symbolizes the standard gamma function and $\sigma^2_{\chi_{2p}} = 2\{p-[\Gamma(p+1/2)/\Gamma(p)]^2\}\sigma^2$. Specifically, the χ_2 or Rayleigh pdf

$$f_{|E_\alpha|}(|e_\alpha|) = 2\,(1-\pi/4)(|e_\alpha|/\sigma^2_{\chi_2})\exp\left[-\,(1-\pi/4)\;|e_\alpha|^2/\sigma^2_{\chi_2}\right] \tag{67}$$

for the magnitude of a single local Cartesian field component α, and the χ_6 pdf

$$f_{|E_t|}(|e_t|) = (3-225\pi/256)^3\,(|e_t|^5/\sigma^6_{\chi_6})\exp\left[-\,(3-225\pi/256)\,|e_t|^2/\sigma^2_{\chi_6}\right] \tag{68}$$

for the magnitude of the total local field are of particular practical relevance [7]. The Rayleigh pdf is related to the Nakagami–Rice distribution describing signal fading and direct illumination [35]. Note that

$$\langle|E_t|\rangle = (15/8)\langle|E_\alpha|\rangle \neq \sqrt{3}\langle|E_\alpha|\rangle. \tag{69}$$

The distributions (62) and (66) have been experimentally verified [6, 7, 34].

All three conditions of isotropy, homogeneity and randomness of polarization must be satisfied in order to obtain a truly random (reverberant) field. For example, for an unpolarized *propagating* plane wave the magnitudes $|\underline{E}|$ and $|\underline{H}|$ each exhibit a χ_4 pdf, i.e. (66) with $p = 2$ and the direction of energy propagation $\underline{1}_k$ is deterministic. In this case ($\underline{E}$, $\underline{H}$, $\underline{k}$) no longer constitutes a triplet of mutually angularly independent random vector variables, for $\underline{E}$ and $\underline{H}$ are confined to the transverse plane ($E_k = H_k = 0$). Consequently $\langle|\underline{E}|\rangle\langle|\underline{H}|\rangle \neq \langle|\underline{E}||\underline{H}|\rangle \neq 0$, and the mean *power* density $\langle\underline{E}\times\underline{H}^*\rangle = \langle|\underline{E}||\underline{H}|\rangle\underline{1}_k$ does no longer vanish, unlike for isotropic (i.e. 3D) random fields for which $\underline{1}_k$ is random with $\langle\underline{1}_k\rangle = \underline{0}$. A plane unpolarized wave is therefore to be considered as an anisotropic (2D) partially random field. By contrast, an unpolarized *standing* plane wave has a binary $\underline{1}_k$ with again $\langle\underline{1}_k\rangle = \underline{0}$ and therefore constitutes a true isotropic random field according to the above definition.

4.3 Wave impedance

By choosing the reference phase for the magnetic field such that $H''_\beta = 0$, the wave impedance $Z = E_\alpha/H_\beta$ with $\alpha, \beta = x, y, z$ is the complex sum of ratios of two Gauss normal random variables for which (27) holds, mutatis mutandis. The distribution of its magnitude $|Z|$ in a perfect reverberant field, where $|H_\beta|$ has the same Rayleigh distribution as $|E_\alpha|$, can be obtained from the transformation $Z = \sqrt{M}$ of the Snedecor $F_{2,2}$ distribution of $M = |E_\alpha|^2/|H_\beta|^2$. With $f_M(m) = 2/(1+m)^2$ we find:

$$\begin{aligned} f_{|Z|}(|z|) &= f_M[M(|z|)] \left| \left(\frac{\mathrm{d}M(|Z|)}{\mathrm{d}|Z|} \right)_{|Z|=|z|} \right| \\ &= \frac{2|z|}{(1+|z|^2)^2}. \end{aligned} \tag{70}$$

This result depends on the statistical independence of E_α and H_β which, with $\underline{E}$ and $\underline{B}$ as the independent source fields, holds for nonmagnetic media ($H_\beta = B_\beta/\mu_0$). The phase angle of the impedance $\Psi_Z = \Psi_\alpha - \Psi_\beta$, with $f_{\Psi_\alpha}(\psi_\alpha) = f_{\Psi_\beta}(\psi_\beta) = (2\pi)^{-1}$, has a triangular pdf:

$$f_{\Psi_Z}(\psi_Z) = \begin{cases} [1 + \psi_Z/(2\pi)]/(2\pi), & -\pi \le \psi_Z \le 0 \\ [1 - \psi_Z/(2\pi)]/(2\pi), & 0 \le \psi_Z \le \pi. \end{cases} \tag{71}$$

5 Complex media in random fields

5.1 Dipolarization and flux density

For linear materials in which the origin of the randomness of $\underline{E}$ is external to the medium, $\underline{P} = X\underline{E}$ is a product of two independent random variables. Consequently, the Cartesian component P_α has as pdf:

$$\begin{aligned} f_{P_\alpha}(p_\alpha) &= \int_{-\infty}^{+\infty} |x|^{-1} f_X(x) f_{E_\alpha}(p_\alpha/x) \mathrm{d}x \\ &= \int_{-\infty}^{+\infty} \left[2\pi\ \sigma_A\ \sigma_{E_\alpha}\ |x|\ (1+\theta x)^2 \right]^{-1} \cdot \\ &\qquad \exp\left\{ -\left[x/(1+\theta x) - \mu_A\right]^2/(2\sigma_A^2) \right. \\ &\qquad\qquad \left. - \left[(p_\alpha/x - \mu_{E_\alpha})^2\right]/(2\sigma_{E_\alpha}^2) \right\} \mathrm{d}x. \end{aligned} \tag{72}$$

Assuming dilute mixtures and mean-centralized X and E_α, the earlier results (9) and (36) yield the pdf for the real and imaginary parts of P_α as:

$$f_{P_\alpha^{\prime(\prime)}}(p_\alpha^{\prime(\prime)}) = (\pi\sigma_X\sigma_E)^{-1} K_0\left[\frac{p_\alpha^{\prime(\prime)}}{\sigma_X\sigma_E}\right]. \tag{73}$$

From (73), the pdf of the corresponding real and imaginary parts of a Cartesian component of the electric flux density D_α follows as $f_{D_\alpha}(d_\alpha) = f_{P_\alpha}(d_\alpha - \epsilon_o\, e_\alpha)$.

The total field E_t, which as the complex sum of three independent circular Gauss normal distributions is also circular Gauss normally distributed with mean $\mu_{E_t} = \sum_{i=1}^{3} \mu_{E_i}$ and Bienaymé variance $\sigma_{E_t}^2 = \sum_{i=1}^{3} \sigma_{E_i}^2$, yields the same distribution as (72) but with all statistics relating to E_t rather than E_α.

5.2 Energy density

Unless $\rho_{P,E}$ is known, the pdf of $S = -\underline{P} \cdot \underline{E}^*$ cannot be computed in the general case, using the product rule for $f_P(p)$ and $f_E(e)$, because $\underline{P}$ and $\underline{E}$ are not independent random variables. Again, however, for linear materials and negligible correlations between the fluctuations of X and those of the internal field, $\underline{P} = X\underline{E}$ so that $S = -X|E|^2$ is a product of independent random variables. Then, for the energy density associated with a Cartesian field component E_α:

$$\begin{aligned} f_{S_\alpha}(s_\alpha) &= \int_{-\infty}^{+\infty} |x|^{-1} f_X(x) f_{|E_\alpha|^2}(s_\alpha/x)\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} \left[\sqrt{2\pi}\ \sigma_A\ \sigma_{|E_\alpha|^2}\ |x|\ (1+\theta x)^2\right]^{-1} \cdot \\ &\qquad \exp\left\{-\left[x/(1+\theta x) - \mu_A\right]^2 / \left(2\sigma_A^2\right)\right. \\ &\qquad\qquad \left. -(s_\alpha/x)/\sigma_{|E_\alpha|^2}\right\} \mathrm{d}x, \end{aligned} \tag{74}$$

whereas for the total field:

$$\begin{aligned} f_{S_t}(s_t) = \int_{-\infty}^{+\infty} &\left[2\sqrt{2\pi}\ \sigma_A\ \sigma_{|E_t|^2}^3\ |x|\ (1+\theta x)^2\right]^{-1} \cdot \\ &\left[3\sqrt{3}\, s_t^2 \exp\left\{-\left[x/(1+\theta x) - \mu_A\right]^2 / \left(2\sigma_A^2\right)\right.\right. \\ &\qquad \left.\left. -\sqrt{3}(s_t/x)/\sigma_{|E_t|^2}\right\}\right] \mathrm{d}x, \end{aligned} \tag{75}$$

which are best solved numerically. If the angular frequency is itself random, then the energy and power densities are no longer proportional to one another, unlike the case for deterministic fields. In this case, unless the time constant of the stirring process is much smaller than the mean period of the wave, the pdf of the random power density $\mathcal{P}$ should be considered, viz. $f_{\mathcal{P}}(p) = \int_{-\infty}^{+\infty} |x|^{-1} f_{\Omega}(x) f_S(p/x) \mathrm{d}x$.

5.3 Wave number

The pdf (41) transforms for random Ω to:

$$f_K(k) = \frac{2}{\pi \sigma_{\mathcal{U}} \sigma_{\mathcal{E}}} \int_{-\infty}^{+\infty} f_{\Omega}(k/x) K_0 \left[\frac{x^2}{\sigma_{\mathcal{U}} \sigma_{\mathcal{E}}} \right] \mathrm{d}x. \tag{76}$$

It is of practical importance, for example, in the characterization of composite media in mode-stirred or mode-tuned reverberation environments, in which complex spatial and temporal field maps are generated inside a cavity by means of electronic stirring (perturbation of operation frequency) or mechanical stirring (rotating paddle wheel) [6, 34]. The changes in electromagnetic boundary configurations give rise to perturbations of the eigenmodes and eigenfrequencies within the cavity bandwidth. Hence the test specimen is excited, statistically, by fields with random frequencies Ω within the stirring bandwidth W. We consider two particular cases of (76). Firstly, assume a uniform (rectangular) pdf $f_{\Omega}(\omega) = w^{-1}$ with $w_c - w/2 < \omega < w_c + w/2$. Then,

$$f_K(k) = \frac{2}{\pi \sigma_{\mathcal{U}} \sigma_{\mathcal{E}}\, w} \int_{k/(w_c+w/2)}^{k/(w_c-w/2)} K_0 \left[\frac{x^2}{\sigma_{\mathcal{U}} \sigma_{\mathcal{E}}} \right] \mathrm{d}x, \tag{77}$$

which approaches $\Gamma^2(1/4)/(\pi\sqrt{2\sigma_{\mathcal{U}}\sigma_{\mathcal{E}}}\, w)$ for $w/k \to +\infty$. Secondly, assume a Cauchy–Breit–Wigner distribution for Ω with scale parameter w, i.e. $f_{\Omega}(\omega) = (\pi w)^{-1} \left[1 + (\omega/w)^2\right]^{-1}$. This is characteristic, for example, of a band-pass filter or single cavity mode of bandwidth W rad/s with centralized Ω whose amplitude characteristic corresponds to $f_{\Omega}(\omega)$. Then, using (6.562.3) of [25], we obtain:

$$f_K(k) \;=\; \frac{2}{\pi^2 \sigma_{\mathcal{U}} \sigma_{\mathcal{E}} w} \int_0^{+\infty} \frac{\sqrt{x}}{x + (k/w)^2} K_0 \left[\frac{x}{\sigma_{\mathcal{U}} \sigma_{\mathcal{E}}} \right] \mathrm{d}x$$

$$
\begin{aligned}
= \quad & \frac{\Gamma^2(1/4)}{\pi^2\sqrt{2\sigma_{\mathcal{U}}\sigma_{\mathcal{E}}}w}\ {}_1F_2\left(1;\frac{3}{4},\frac{3}{4};\frac{(k/w)^4}{4\sigma_{\mathcal{U}}^2\sigma_{\mathcal{E}}^2}\right) \\
& - \frac{2}{\pi\sigma_{\mathcal{U}}\sigma_{\mathcal{E}}w}K_0\left[\frac{(k/w)^2}{\sigma_{\mathcal{U}}\sigma_{\mathcal{E}}}\right]\ (k/w) \\
& - \frac{\Gamma^2(-1/4)}{\pi^2(2\sigma_{\mathcal{U}}\sigma_{\mathcal{E}})^{3/2}w}\ {}_1F_2\left(1;\frac{5}{4},\frac{5}{4};\frac{(k/w)^4}{4\sigma_{\mathcal{U}}^2\sigma_{\mathcal{E}}^2}\right)\ (k/w)^2,
\end{aligned}
\tag{78}
$$

where ${}_1F_2(\cdot;\cdot,\cdot;\cdot)$ is the generalized Barnes hypergeometric series. An approximate solution of (77) can be obtained equivalently by taking (78) in which w is replaced by $(\pi/2)w$ [36].

5.4 Current density

For time-harmonic excitation, the pdfs of the in-phase component $J' = \Omega\mathcal{E}''E \triangleq \Sigma E$ and quadrature component $J'' = \Omega\mathcal{E}'E$ of $j\Omega D$ are, using (36):

$$
f_{J'^{(\prime)}}(j'^{(\prime)}) = (\pi\,\sigma_{\mathcal{E}^{(\prime)\prime}}\,\sigma_E)^{-1}\int_{-\infty}^{+\infty}|t|^{-1}f_\Omega(t)K_0\left[\frac{|j'^{(\prime)}/t|}{\sigma_{\mathcal{E}^{(\prime)\prime}}\,\sigma_E}\right]\mathrm{d}t. \tag{79}
$$

6 Conclusions

We have shown how a complete first-order statistical electromagnetic characterization of random composite media can be performed when accounting for the uncertainties on the characteristics of the medium or fields. From the pdfs follow statistics for various macroscopic quantities. The analysis depends critically on the basic assumptions for the 'fundamental' characteristics of the medium, which have been given a physical basis. These assumptions yield underlying parent distributions based on the central limit theorem. The incoporation of deviations for these pdfs to include skewness, noncentrality, correlation, etc. requires more elaborate parent pdfs.

On some occasions in our exposition, the lack of statistical independence between two random variables prevented us from rigorously expressing the pdf of their product or ratio. The difficulty may be resolved, at least partially, by applying a transformation of

these variables to their principal components (orthogonal expansion), which eliminates their first-order correlation i.e. their *linear* dependence. This procedure has been detailed in e.g. [31]. Only in a limited number of cases, viz. if the functional form of the correlation function yields a kernel that is integrable, can this transformation be performed analytically; otherwise a numerical (matrix) computation is required to solve the associated eigenvalue problem. If the original variables exhibit a Gauss normal pdf, the procedure leads to an exact solution because of the absence of higher-order correlations (*statistical* independence). For example, in (27) the Kramers–Kronig-correlated $\mathcal{E}'$, $\mathcal{E}''$ may first be transformed to the eigenvalues Λ', Λ'' of the covariance $\langle(\mathcal{E}' - \langle\mathcal{E}'\rangle)(\mathcal{E}'' - \langle\mathcal{E}''\rangle)\rangle$, whence $f_{\mathrm{Tan}\Delta}(\tan\delta)$ is obtained upon subsequent inverse transformation of $f_{\Lambda''/\Lambda'}(\lambda''/\lambda')$.

An extension to higher-order multipole moments is in order, because multipoles give rise to peculiar effects at boundaries between different media in general, and at interfaces between a particle and the host medium in particular [13, Section III.8.3(ii)]. This is because volume integrals over the multi-divergence of the macroscopic electric multipole moment densities, as part of the total electric polarization, are equivalent to double integrals for the surface charge density on surfacial multi-layers.

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